Microeconomics
Solution/General Equilibrium

Exercise 1

Consider a two-individual, two-good economy. The preferences of individual A and individual B over good 1 and good 2 are represented by the utility function:

$$u(x_1, x_2) = x_1 x_2, \ x_1 \geq 0, x_2 \geq 0,$$

where x_i denotes the quantity of good i consumed by the individual. There are 3 units of each good in the economy.

1. **Characterize the set of Pareto optimal allocations.**
   The agents have the same Cobb-Douglas preferences over consumption bundles. Interior Pareto optimum allocations are characterized by

$$MRS_{12}^{A}\left(x_1^A, x_2^A\right) = MRS_{12}^{B}\left(x_1^B, x_2^B\right) \Leftrightarrow \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}. \tag{1}$$

   Using the binding material constraints,

$$\begin{aligned} x_1^B &= 3 - x_1^A, \\ x_2^B &= 3 - x_2^A, \end{aligned}$$

   (1) is equivalent to

$$\frac{x_2^A}{x_1^A} = \frac{3 - x_2^A}{3 - x_1^A} \Leftrightarrow x_2^A\left(3 - x_1^A\right) = x_1^A\left(3 - x_2^A\right) \Leftrightarrow x_1^A = x_2^A,$$

   which corresponds to the 45°-line inside the Edgeworth box. We have to add $(0,0)$ and $(3,3)$ to this segment to get the whole Pareto set:

$$\left\{\left(x_1^A, x_2^A, x_1^B, x_2^B\right) \in [0,3]^4 : x_1^A = x_2^A \text{ and } x_1^B = x_2^B = 3 - x_1^A\right\}.$$

The 3 units of each good are initially shared as follows between the two individuals: A has one unit of good 1 and 2 units of good 2; B has 2 units of good 1 and 1 unit of good 2.

2. **Determine the Walrasian equilibria.**

A Walrasian equibrium is (x^*, p^*) such that: (1) each individual maximizes his utility s.t. his budget constraint; (2) the market clearing condition holds.
Individual $i's$ UMP leads to the Marshallian demand functions:

$$x_1^i(p, w) = \frac{1}{2}\frac{p_1 + 2p_2}{p_1} \text{ and } x_2^i(p, w) = \frac{1}{2}\frac{p_1 + 2p_2}{p_2}, \ i = \{A, B\}.. \tag{2}$$

By Walras' law, each markets clears iff:

$$2x_1^i(p,w) = 3 \Leftrightarrow \frac{p_1 + 2p_2}{p_1} = 3 \Leftrightarrow \frac{p_1}{p_2} = 1. \tag{3}$$

The Walrasian equilibrium is therefore $\left(x_1^A, x_2^A, x_1^B, x_2^B\right)^* = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$ and $p_1 = p_2$.
At the equilibrium, the resources of the economy are shared equally.

3. **Represent the economy in the Edgeworth box. Comment.**
   In this economy, everything is symmetric. So, at the Walrasian equilibrium, the Marshallian demands for goods 1 and 2 of every individual must be the same. Consequently, the Walrasian equilibrium must be at the center of the Edgeworth box $\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$.

Exercise 2

Consider a two-person, two-good pure exchange economy. A's preferences over consumption bundles (x_1, x_2) are represented by the utility function

$$U^A(x_1, x_2) = x_1 x_2$$

where $x_1 \geq 0$ denotes the quantity of good 1 and $x_2 \geq 0$ the quantity of good 2. B's preferences over consumption bundles are represented by the utility function

$$U^B(x_1, x_2) = x_1 + x_2.$$

The initial endowments in goods 1 and 2 are respectively $\Omega^1 = 1$ and $\Omega^2 = 2$.

1. Represent this economy in the Edgeworth box.
2. Define a Pareto optimum allocation for this economy.
3. Derive the set of Pareto optimum allocations. Represent it on the graph depicted in Question 1.
4. Would you have enough information to compute the general equilibria of this economy? Explain the difference between a Pareto optimal allocation and a general equilibrium.

Paeto optimal allocations are solution to

$$\max_{x^A \geq 0, x^B \geq 0} U^A\left(x_1^A, x_2^A\right) \tag{4}$$

subject to

$$U^B\left(x_1^B, x_2^B\right) \geq \overline{u}^B, \tag{5}$$

$$0 \leq x_1^A + x_1^B \leq 1, \tag{6}$$

$$0 \leq x_2^A + x_2^B \leq 2. \tag{7}$$

The optimization programme is concave. The material balance constraint are binding at the optimum. So, the Lagragian can be written as:

$$L = x_1^A x_2^A + \lambda\left(x_1^B + x_2^B\right) = x_1^A x_2^A + \lambda\left(1 - x_1^A + 1 - x_2^A\right). \tag{8}$$

The necessary and sufficient first-order conditions are:

$$x_2^A - \lambda \leq 0 \left(= 0 \text{ if } x_1^A > 0\right), \tag{9}$$
$$x_1^A - \lambda \leq 0 \left(= 0 \text{ if } x_2^A > 0\right). \tag{10}$$

There are three cases.

Case 1: Interior solution. Obtained when $x_1^A > 0$ and $x_2^A > 0$. The first-order conditions are:

$$\frac{x_2^A}{x_1^A} = 1 \Leftrightarrow x_2^A = x_1^A. \tag{11}$$

Therefore,

$$x_1^B = 1 - x_1^A, \tag{12}$$
$$x_2^B = 1 - x_2^A. \tag{13}$$

Case 2: Corner solution with $x_1^A = 0$. Since A has Cobb-Douglas preferences which do not intersect the $x_1^A = 0$ axe for $x_2^A > 0$, it must be $x_2^A = 0$. A's origin belongs to the Pareto set. (Have a look at the figure drawn in Question 1)

Case 3: Corner solution with $x_1^A = 1$. For $x_2^A < 1$, there is no maximum with $x_1^A = 0$. For $x_2^A \geq 1$, and $U^B = \bar{u}_B$ given, U^A is maximum along the right-hand side of the Edgeworth rectangle. (Have a look at the figure drawn in Question 1)

In summary, the Pareto set is given by $0 \leq x_1^A \leq 1, 0 \leq x_2^A \leq 2$ such that:

$$x_1^A = x_2^A, \tag{14}$$
$$\text{or } x_1^A = 1 \text{ and } 1 \leq x_2^A \leq 2. \tag{15}$$

Exercise 3

Consider a two-person two-good pure exchange economy. Person A's and person B's preferences over consumption bundles $(x_1, x_2) \in \mathbb{R}_+^2$ are represented by the following utility functions:

$$U^A(x_1, x_2) = x_1 + \ln x_2,$$
$$U^B(x_1, x_2) = x_1 + 2\ln x_2.$$

There are initially 5 units of good 1 and 3 units of good 2 in the economy.

1. **Determine the set of Pareto optimum allocations. Represent it in the Edgeworth box.**
   Interior Pareto optimum allocations are characterized by

$$MRS_{12}^A\left(x_1^A, x_2^A\right) = MRS_{12}^B\left(x_1^B, x_2^B\right) \Leftrightarrow x_2^A = \frac{x_2^B}{2}. \tag{16}$$

   Using the material constraint $x_2^A + x_2^B \leq 3$, which must be binding at any Pareto optimum since U^A and U^B are strictly increasing in x_1 and x_2, (16) is equivalent to

$$x_2^A = \frac{3 - x_2^A}{2} \Leftrightarrow x_2^A = 1.$$

   Hence, interior Pareto optimum allocations are such that

$$x_2^A = 1 \text{ and } x_2^B = 2.$$

The Pareto set is therefore

$$\left\{\left(x_1^A, x_1^B, x_2^A, x_2^B\right) \in (0,5)^2 \times (0,3)^2 : x_2^A = 1 \text{ and } x_2^B = 2\right\} \cup \{\{0\} \times [0,1]\} \cup \{\{5\} \times [1,2]\}.$$

2. Consider any Pareto optimum allocation $\left(x_1^A, x_2^A, x_1^B, x_2^B\right) >> 0$, where x_i^j stands for the quantity of good i available to person j.

   (a) **Establish that it is a general equilibrium associated with a price vector and a distribution of person *A*'s and person *B*'s exogenous incomes, denoted ω^A and ω^B respectively. Without loss of generality, the price of good 1 will be normalized to 1.**
   It must be established that there is a price vector and a distribution of wealth for which any given interior Pareto optimum is obtained as a general equilibrium. To this aim, let us first derive each consumer's Marshallian demand functions.
   (i) Since we are focusing on interior Pareto optimum allocations, we are only interested in the demand functions for interior solutions. A necessary condition for A's utility maximization programme is:

$$\left|MRS_{12}^A\left(x_1^A, x_2^A\right)\right| = x_2^A = \frac{1}{q},$$

   where $p = 1$ is the normalized price of good 1 and q the price of good 2. Using A's budget constraint, which must be binding, one gets:

$$x_1^A = \omega^A - qx_2^A = \omega^A - 1,$$

   where ω_A is A's initial wealth.
   Proceeding similarly, it is found that

$$\left|MRS_{12}^B\left(x_1^A, x_2^A\right)\right| = \frac{x_2^B}{2} = \frac{1}{q} \Leftrightarrow x_2^B = \frac{2}{q},$$

   from which

$$x_1^B = \omega_B - qx_2^B = \omega_B - 2.$$

   (ii) Any interior Pareto optimum allocation must satisfy

$$x_2^A = 1 \text{ and } x_2^B = 2.$$

   Therefore, using (i), the utility maximization programmes of the consumers lead to an interior Pareto optimum allocation provided $1/q = 1$ and $2/q = 2$, i.e.

$$q = 1,$$

   in which case

$$\begin{aligned} \omega^A &= x_1^A + 1, \\ \omega^B &= x_1^B + 2. \end{aligned}$$

   (b) **Determine the Pareto optimum allocation for which the distribution of incomes is egalitarian in the sense that $\omega^A = \omega^B$.**
   Using Question 2b, it must be

$$\omega^A + \omega^B = x_1^A + x_1^B + 3 \Leftrightarrow 2\omega^A = 8 \Leftrightarrow \omega^A = 4.$$

So, $\omega^A = \omega^B = 4$ with $q = 1$, from which

$$\begin{aligned} x_1^A &= \omega^A - 1 = 3, \\ x_2^A &= 1, \\ x_1^B &= \omega_B - 2 = 2, \\ x_2^B &= 2. \end{aligned}$$

(c) **Person A has 4 units of good 1 and 2 units of good 2. Person B has 1 unit of each good. What are the income transfers from A to B which lead to the egalitarian equilibrium?**
Given prices $p = q = 1$, A's initial wealth amounts to

$$\omega^A = 6.$$

Let T denote a transfer from A to B. Then, A's post-transfer wealth is

$$\omega^A - T = 6 - T.$$

B's post-transfer wealth is thus

$$\omega^B + T = 2 + T.$$

We want to equalize post-transfer wealth, i.e. determine T such that

$$6 - T = 2 + T \Leftrightarrow T = 2.$$

Exercise 4

Consider a two-good (1 and 2), two-individual (A and B) economy. Let $x = (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ be a consumption bundle. A's preferences are represented by the utility function defined over consumption bundles, $u_A : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ with

$$u_A(x) = x_1 + \ln x_2,$$

B's preferences are represented by $u_B : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ with

$$u_B(x) = \min\{x_1, x_2\}.$$

A and B's endowments in goods 1 and 2 are $\omega^A = (3, 0)$ and $\omega^B = (0, 3)$.

1. **Compute the Walrasian equilibrium.**
(i) Draw a graph to see that all Pareto optimal allocations are interior (except 0_A and 0_B). Since each Walrasian equilibrium is Pareto optimal, the equilibrium will be in the interior of the Edgeworth box and $x_1^i = x_2^i$, $i = A, B$ at this equilibrium.
(ii) B's utility maximization programme yields

$$x_1^B = x_2^B = \frac{w^B}{p_1 + p_2}. \quad (17)$$

A's utility maximization programme is

$$\max x_1 + \ln x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 = w_A, \quad (18)$$

from which

$$x_2^A = \begin{cases} \frac{p_1}{p_2} \text{ if } w_A \geq p_1, \\ w_A / p_2 \text{ otherwise.} \end{cases} \quad (19)$$

Here, $w^A = 3p_1 > p_1$, so the solution is always interior.
(iii) Market clearing requires:

$$\frac{p_1}{p_2} + \frac{3p_2}{p_1 + p_2} = 3 \Leftrightarrow 0 = p(2-p) \Leftrightarrow p = 2, \text{ with } p = \frac{p_1}{p_2}. \tag{20}$$

The unique equilibrium is thus:

$$E^1 = \left(p, x_1^A, x_2^A, x_1^B, x_2^B\right)^* = (2, 2, 2, 1, 1). \tag{21}$$

2. A productive sector is now added to the pure exchange economy considered above. The productive sector consists of one firm which transforms a quantity y_1 of good 1 in a quantity y_2 of good 2 according to the technology
$$y_2 = f(y_1) = ay_1, \text{ where } a > 0.$$

   (a) **Define a Walrasian equilibrium for this economy.**
   Given a private economy preferences and endowments a (Walrasian) equilibrium is (x, y, p) s.t.
   (i) the firm chooses its production levels so as to maximize its profits;
   (ii) A and B maximize their respective utility subject to their budget constraint;
   (iii) market clear, i.e. $x_2^{A*} + x_2^{B*} = 3 + y_2$.

   (b) **Compute a Walrasian equilibrium E^2 for this economy. Comments?**
   The firm has a constant-returns-to-scales technology. The firm's profit is

$$\pi = p_2 y_2 - p_1 y_1 = (p_2 a - p_1) y_1.$$

   (i) *Let us suppose there is a strictly positive quantity solution to the firm's maximisation programme.* Then, it must be

$$ap_2 - p_1 = 0 \Leftrightarrow \frac{p_1}{p_2} = a. \tag{22}$$

   Without loss of generality, let $p_1 = 1$. Then

$$x_2^A = a \text{ and } x_2^B = \frac{3}{a+1}. \tag{23}$$

   By Walras' law, the production plan $(-y_1, ay_1)$ is chosen so as to clear the market for good 2:

$$a + \frac{3}{a+1} = 3 + ay_1 \Leftrightarrow y_1 = \frac{a-2}{1+a}. \tag{24}$$

   In conclusion,

$$a > 2 : y_1^* = \frac{a-2}{1+a} \text{ and } y_2^* = \frac{a(a-2)}{1+a}, \text{ etc.} \tag{25}$$
$$0 < a \leq 2 : y_1^* = y_2^* = 0 \text{ and } E^2 = E^1. \tag{26}$$

   If the technology used to transform good 1 into good 2 is not effective enough, the firm won't produce at the general equilibrium.

Exercise 5

Let us consider a two-good, one-consumer (A) economy. x is the sole consumption good and leisure is denoted L. A's preference relation is represented by the utility function $U(x,L)=x^{\alpha}L^{1-\alpha}$, with $0<\alpha<1$. His initial endowment is $\left(0,\overline{L}\right)$. So, labour is defined as $\ell:=\overline{L}-L$. A firm uses labour ℓ to produce the consumption good x.

1. **Examine the Walrasian equilibria of this economy when the technology is** $x=\ell^{1/2}$.

   Without loss of generality, the price of the consumption good is normalized to 1.
   (i) The firm's profit is $\pi=\sqrt{\ell}-\omega\ell$. The FOC is:

   $$\frac{1}{2}\ell^{-1/2}-\omega=0\Leftrightarrow\ell^{d}(\omega)=\frac{1}{(2\omega)^{2}}. \tag{27}$$

   So, the firm's supply is

   $$x^{s}=\frac{1}{2\omega}. \tag{28}$$

   The firm's profit amounts to:

   $$\pi^{*}(\omega)=\frac{1}{2\omega}-\omega\frac{1}{(2\omega)^{2}}=\frac{1}{4\omega}. \tag{29}$$

   (ii) The consumer chooses ℓ to maximize

   $$U(x,\ell)=x^{\alpha}\left(\overline{L}-\ell\right)^{1-\alpha}\text{ s.t. }x+\omega\left(\overline{L}-\ell\right)=\overline{L}\omega+\pi^{*}(\omega).$$

   Hence,

   $$\overline{L}-\ell=(1-\alpha)\frac{\overline{L}\omega+\pi^{*}(\omega)}{\omega}=(1-\alpha)\left[\overline{L}+\frac{1}{4\omega^{2}}\right]\Leftrightarrow\ell^{s}(\omega)=\overline{L}\alpha-\frac{1-\alpha}{4\omega^{2}}. \tag{30}$$

   By Walras' law, the wage equilibrium is obtained for $\ell^{s}(\omega)=\ell^{d}(\omega)$, i.e.

   $$\omega^{*}=\sqrt{\frac{1}{\overline{L}}\frac{2-\alpha}{4\alpha}}>0. \tag{31}$$

2. **Examine the Walrasian equilibria of this economy when the technology is** $x=a\ell$, $a>0$.
   We know that each factor is paid at its marginal productivity. Hence,

   $$\omega^{*}=a.$$

   In addition,

   $$\pi^{*}=0.$$

   So, A's budget constraint reads:

   $$x+\omega\left(\overline{L}-\ell\right)=\overline{L}\omega. \tag{32}$$

   Since the considered individual has Cobb-Douglas preferences, his Marshallian demand functions are

   $$\begin{aligned}\overline{L}-\ell&=(1-\alpha)\frac{\overline{L}\omega}{\omega},\\ x&=\alpha\overline{L}\omega.\end{aligned}$$

(Recall that the consumption price has been normalized to $1)$. Consequently, at the equilibrium,

$$\left(\overline{L}-\ell\right)(\omega^*) = (1-\alpha)\overline{L} \Leftrightarrow \ell(\omega^*) = \overline{L}\alpha, \tag{33}$$

$$x(\omega^*) = \omega^*\ell = \overline{L}a\alpha. \tag{34}$$

With constant returns to scale, there is a *dichotomy* when solving for the equilibrium: the production side of the economy yields the price equilibrium, while the consumption side of the economy provides the quantities which are produced and traded at the equilibrium.

3. The technology is $x = \ell^2$.

   (a) **Represent the production possibility set and draw some indifference curves in the (ℓ, x)-space.**
   For $\alpha = 1/2$, for instance, the equation of the indifference curve of level k is:

   $$x = \frac{e^{2k}}{\overline{L}-\ell}, \tag{35}$$

   with $x' = \frac{e^{2k}}{\left(\overline{L}-\ell\right)^2} >$ and $x'' = 2\frac{e^{2k}}{\left(\overline{L}-\ell\right)^3} > 0$.

   (b) **Determine the Pareto optimal allocation of this economy. Represent it in the same graph.**
   The Pareto optimal allocation is solution to:

   $$\max_{x,\ell} U\left(x, \overline{L}-\ell\right) \text{ s.t. } x = \ell^2,\ 0 \leq \ell \leq \overline{L}. \tag{36}$$

   This programme is equivalent to maximizing $2\alpha \ln \ell + (1-\alpha)\ln\left(\overline{L}-\ell\right)$ s.t. $0 \leq \ell \leq \overline{L}$. Assuming an interior solution, the FOC is:

   $$\ell^* = \overline{L}\frac{2\alpha}{1+\alpha}. \tag{37}$$

   Since $\ell > 0$ and

   $$\frac{2\alpha}{1+\alpha} < 1 \Leftrightarrow \alpha < 1, \tag{38}$$

   the solution is interior (provided $\alpha < 1)$. The production x^* is thus:

   $$x^* = \left(\overline{L}\frac{2\alpha}{1+\alpha}\right)^2. \tag{39}$$

   (ℓ^*, x^*) is Pareto optimal.

   (c) **Show that the Pareto optimal allocation cannot be decentralized as a Walrasian equilibrium. Comment.**
   Let us try to decentralize this allocation. The consumer owns all initial endowments (i.e. a quantity of labour). His budget constraint reads:

   $$x + \omega\left(\overline{L}-\ell\right) = \overline{L}\omega + \pi, \tag{40}$$

   where π is the firm's profit. Hence, the consumer's labour supply amounts to

   $$\ell = \overline{L}\alpha - \frac{(1-\alpha)\pi}{\omega}. \tag{41}$$

If ℓ is equalized to ℓ^*, the Pareto optimum labour supply found in (b),

$$\pi = -\omega \frac{\overline{L}\alpha}{1+\alpha} < 0. \tag{42}$$

The equilibrium can be decentralized only if $\pi < 0$.
As a consequence, this Pareto optimal allocation cannot be decentralized with a wage earner, on the one hand, and an entrepreneur, on the other hand (provided the entrepreneur's preference relation is monotonic and thus his utility function increasing in π, which is the case for a rational agent). *The second fundamental theorem of welfare economics cannot apply because of the non-convexities of the production side of the economy.*