4. Game Theory: Introduction

Laurent Simula

ENS de Lyon

- Textbook : Prajit K. Dutta, Strategies and Games, Theory and Practice, MIT Press, 1999

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Game theory

- Deals with all real-life situations where people *interact* with each other.
- By interaction, we mean that the outcome for a single player depends on the decisions made by the other players.

Game theory has a very large scope

- In political science: parties competing for office, lobbying...
- In economics: oligopolistic competition (Carrefour, Géant, Monoprix, etc.), firm owners whose outcome depends on the employees' effort, tax competition between Nations...
- In computer science: P2P and torrent protocols to share data (e.g. you won't be able to download if you do not share contents)
- In international relations: cold war, decision making within the UN security council...
- In education: students competing for scholarships, students involved in a joint project (the outcome depending on total effort, a few students can decide to "slack")...

Rational behaviour

We will assume that players are rational:

- Their choices are made consistently.
- Given all relevant pieces of information at his/her disposal, a consistent player acts so as to maximise his/her payoff.

Non-cooperative games

We will focus on non-cooperative game theory.

- Players' decisions only based on perceived self-interest (but somebody's self-interest may be to maximise the payoff of someone else).
- Players' commitments not enforceable (except those allowed by the rules of the game).
- Cooperation is possible but only if it is self-enforceable (i.e. in the interest of each player and no player has an incentive to break it when it is implemented by the other parties).

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Experiment #1: the "grade game"

"Without showing your neighbour what you are doing, write down on a form either the letter α or the letter β (and also your name). Think of this as a 'grade bid'. I will collect all the forms and randomly pair your form with one other form. Neither you nor your pair will ever know with whom you were paired. Here is how grades may be assigned for this course.

- If you put α and your pair puts β, then you will get "Distinction", and your pair "Fail".
- If both you and your pair put α, then you both will get "Pass".
- If you put β and your pair puts α, then you will get grade "Fail", and your pair grade "Distinction".
- If both you and your pair put β, then you will both get "Honors".

Note that the grades are ordered as follows: distinction, honors, pass and fail, from the highest to the lowest grade.

A first representation of the game

The possible choices α and β are called pure strategies.

Me

Me\Pair	α	β
α	Pass	Distinction
β	Fail	Honors

my grades

My pair

Me\Pair	α	β
α	Pass	Fail
β	Distinction	Honors

pair's grades

We can use the outcome matrix to represent the game:

Me\Pair	α	β
α	Pass,Pass	Distinction,Fail
β	Fail,Distinction	Honors,Honors

Why did you play what you played?

Possible reasons:

- You prefer Distinction to Honors, Honors to Pass and Pass to Fail.
- You expect that the other player thinks in the same way: he/she prefers Distinction to Honors, Honors to Pass and Pass to Fail. [but after all you are not sure about that]
- Assuming that the other player ranks the alternatives as you do, we obtain the payoff matrix:

Me\Pair	α	β
α	$0, 0$	$2, -1$
β	$-1, 2$	$1, 1$

($\neq$ outcome matrix)

- Note that the numbers (-1, 0, 1, 2) do not really matter per se; what is important is that $-1 < 0 < 1 < 2$, i.e. the ranking of opportunities.

What should a rational player choose?

To answer this question, we need to know what information is available to the players.

Example

In the grading game, you might be wrong when assuming that the other player is as "greedy" as you are. E.g. some people suffer from inequity aversion (either disadvantageous or advantageous) (See Fehr and Smith 1999).

Your pair is as "greedy" as you

Me\Pair	α	β
α	$0, 0$	$2, -1$
β	$-1, 2$	$1, 1$

Your pair is an indignant angel

Me\Pair	α	β
α	$0, 0$	$2, -2$
β	$-1, -1$	$1, 1$

It is important to know whether the payoffs, i.e., individual preferences, are common knowledge. You may know for sure that the other player is "greedy" or benevolent, or that 90% of the students in the class are greedy and 10% benevolent, etc.

What is a game?

To describe strategic interactions, we need to know:

1. The set of players: Who is involved?
2. The rules of the game: What can the players do? Who moves when? And what do they know when they move?
3. The outcomes: For each possible set of actions by the players, what is the outcome of the game?
4. The payoffs: Players' preferences over the possible outcomes (expressed in terms of utility or money).

In the grading game

- Set of players: 1 and 2;
- Actions available to each of them: $\{\alpha,\beta\}$
- Payoff functions given in the payoff matrix: (assuming that my Pair as the same preferences as me)

Me\Pair	α	β
α	$0, 0$	$2, -1$
β	$-1, 2$	$1, 1$

Information

Perfect and imperfect information say something about the **rules** of the game.

- Perfect information: players know exactly what happened in previous moves and there are no simultaneous moves.
- Imperfect information: otherwise.

Complete and incomplete information say something about the **circumstances** of the game.

- Complete information: each element of the game is common knowledge. In particular, every player knows the payoffs and strategies available to other players.
- Incomplete information: otherwise.

In the grading game

Information is extremely important!

- Perfect information? no, because simultaneous moves. $\Rightarrow$ imperfect information.
- Complete information? each player knows the strategies available to the other player but is not a priori completely sure about the preferences of the player with whom he/she will be matched. Hence, incomplete information. If you are sure about the other player's preferences, then complete information.

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Extensive-form games

The extensive-form representation of a game specifies:

- The set of players;
- Whose turn it is to move;
- What every player can choose;
- What he/she knows when he/she takes his/her decision;
- His/her payoff for each combination of moves that could be chosen.

Graphic representation = tree.

Experiment #2: a "nim game"

- 20 objects on the table.
- Player A removes 1, 2 or 3 objects.
- Player B observes Player A's move and, then, removes 1, 2 or 3 objects.
- Player A observes Player B's move and, then, removes 1, 2 or 3 objects, etc.
- **Winner = Player who removes the last object from the table.**
- Players prefer to win than to lose.

Let's play the game!

What is the best strategy?

Let us examine the same game with only 4 objects.

- Draw the tree of the game.
- Examine best actions for every player.
- What is your conclusion?

Game's tree

An extensive-form game is represented by a rooted tree.

- The game starts at the root of the tree.
- We call <u>decision nodes</u> nodes at which a player makes a decision.
- The tree's <u>terminal nodes</u> corresponds to the end points of the game. Each end point is associated with the players' payoffs of that play. [there is no decision to be made at a terminal node.]

In addition, we have to **show which information is available to the players**.

Information sets

- Information set: for a particular player, establishes all the possible moves that could have taken place in the game so far, given what that player has observed so far.
- More specifically, set of decision nodes such that:
  - every node in the set belongs to the **same** player; [an information set is player-specific]
  - when play reaches an information set, the player to whom it belongs does not know which node in the set has been reached. [the player cannot "discriminate" between nodes of a same information set – in the sense that he does not know where he is in the information set]

Games of perfect information $\Longrightarrow$ every information set contains only one member.

Matching pennies

- Two players, 1 and 2:
  - Each puts down a penny: either heads up (H) or tails up (T).
  - If the pennies match, i.e., (H,H) or (T,T), player 1 pays one dollar to player 2, otherwise player 2 pays player 1 one dollar.
  - Utilities measured in money.
- Draw the game tree for these four alternatives:

1. "STANDARD" MATCHING PENNIES: the players play simultaneously.
2. MATCHING PENNIES I: Player 1 moves first and hides his action.
3. MATCHING PENNIES II: Player 1 moves first, player 2 observes player 1's action.
4. MATCHING PENNIES III: They flip a symmetric coin to determine the order of moves. The follower observes the leader's action.

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Strategy: definition and first example

Definition

A player's strategy is a complete specification of how a player intends to play **in every contingency that might arise**.

Example

There are two possible states of the world: it rains today or it does not. There are two possible actions: I take my umbrella or I do not take it. A strategy specifies what I will do if it rains as well as what I will do if it does not rain. In this example, a strategy is therefore a pair of actions.

Strategy: definition and second example

Definition

A player's strategy is a **complete plan of actions**, specifying what the player intends to play in every contingency that might arise.

Example

Strategies in MATCHING PENNIES II?

- Player 1 (leader): 2 possible actions, Heads and Tails. 2 strategies: , Heads and Tails
- Player 2 (follower): 2 possible actions, Heads and Tails. But 4 strategies:
  1. "HH": play H if P1 plays H; play H if P1 plays T;
  2. "HT": play H if P1 plays H; play T if P1 plays T;
  3. "TH": play T if P1 plays H; play H if P1 plays T;
  4. "TT": play T if P1 plays H; play T if P1 plays T.

Strategic form

Definition

A strategic-form representation of a game specifies:

- Set of players N;
- Strategy sets (or action spaces), X_n for $n \in N$.
- Payoffs function $u_n : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ for every player and every possible combination of strategies.

Strategic form: example (1.)

Example

MATCHING PENNIES II.

- Set of players $\{1, 2\}$;
- Strategy sets: $X_1 = \{H, T\}$ for Player 1 and $X_2 = \{HH, HT, TH, TT\}$ for Player 2.
- All possible combinations of (pure) strategies? Cartesian product $X_1 \times X_2$, i.e. (H, HH), (H, HT), (H, TH), (H, TT), (T, HH), (T, HT), (T, TH) and (T, TT).
- Payoff functions $u_1 : X_1 \times X_2 \longrightarrow \mathbb{R}$ for player 1 and $u_2 : X_1 \times X_2 \longrightarrow \mathbb{R}$ for player 2 with:
  - $u_1(s_1, s_2) = +1$ if "$s_1 = H$ and $s_2 = TH$ or TT" and if "$s_1 = T$ and $s_2 = HH$ or TH",
  - $u_1(s_1, s_2) = -1$ if "$s_1 = H$ and $s_2 = HH$ or HT" and if "$s_1 = T$ and $s_2 = HT$ or TT",
  - $u_2(s_1, s_2) = -u_1(s_1, s_2)$.

Easier to summarize the payoffs in the payoff matrix.

Strategic form: example (2.)

MATCHING PENNIES II, payoff matrix:

$1\backslash 2$	HH	HT	TT	TH
H	$-1, 1$	$-1, 1$	$1, -1$	$1, -1$
T	$1, -1$	$-1, 1$	$-1, 1$	$1, -1$

Other example: rock, scissors and paper

- 2 players, who move simultaneously;
- three actions: rock (R), scissors (S) and paper (P);
- R beats S, S beats P and P beats R;
- if same actions, payoff is zero; otherwise the winner gets one dollar from the looser.

Draw the game matrix and the game tree. (Try it on your own!)

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Experiment #3: repeating matching pennies

- Two players, P1 and P2:
  - simultaneously put down a penny: either heads up (H) or tails up (T).
  - If the pennies match, i.e., (H,H) or (T,T), player 1 pays one dollar to player 2, otherwise player 2 pays player 1 one dollar.
- Repeat the game 10 times (see intruction sheet). Then:
  - examine the strategies of each player;
  - compute each player's payoff.
- Discussion. [In particular, what would happen if you were always playing the same action all the time? Benefit of making one's actions unpredictable.]

Pure vs. mixed strategies

- A player may have the possibility to randomize. A given action will then be a probability distribution over the set of pure strategies (as defined above).
- Example: In matching pennies, the action sets are $\{H, T\}$ for both players.
  - Two pure strategies: "playing H" and "playing T";
  - from which mixed strategies can be defined: playing H with probability p and playing T with probability $(1-p)$ with p varying in $[0, 1]$.

Mixed strategies in matching pennies (1.)

- Action sets when allowing for mixed strategies: denoted ΔX_1 and ΔX_2 (instead of X_1 and X_2).
- Payoff functions $u_i : \Delta X_1 \times \Delta X_2 \longrightarrow \mathbb{R}$ with

$$\begin{aligned} & u_1\left(\left(p, 1-p\right), \left(q, 1-q\right)\right) \\ & \quad = \left(-1\right) pq + 1p\left(1-q\right) + 1\left(1-p\right) q + \left(-1\right)\left(1-p\right)\left(1-q\right) \\ & \quad\quad = 2p\left(1-2q\right) + 2q - 1 \end{aligned}$$

and

$$\begin{aligned} u_2\left(\left(p, 1-p\right), \left(q, 1-q\right)\right) &= -u_1\left(p, 1-p, q, 1-q\right) \\ &= 2q\left(2p-1\right) + 1 - 2p. \end{aligned}$$

Mixed strategies in matching pennies (2.)

- What is the best response of Player 1 to Player 2's mixed strategy $(q, 1-q)$?
- What is the best response of Player 2 to Player 1's mixed strategy $(p, 1-p)$?
- For which (p, q) is p a best response to q and q a best response to p? In that case, when $((p, 1-p), (q, 1-q))$ is reached, no one has an incentive to deviate unilaterally. We will see in the next chapter that the pair of mixed strategies $((p, 1-p), (q, 1-q))$ is a Nash equilibrium of the game.