7. Static Games of Incomplete Information

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Setting
- Bayesian Games
- An Example

Normal-Form Representation of Static Bayesian Games
- Types
- Beliefs
- Normal Form
- Timing
- Bayes's rule

Bayesian Nash Equilibrium
- Definition
- Application
- Battle of the Sexes with Two Types
- Pure-Strategy BNE
- Mixed-Strategy BNE

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Setting

Focus on games with incomplete information $\Longrightarrow$ BAYESIAN GAMES.

- COMPLETE INFORMATION: the players' payoff functions are common knowledge.
- INCOMPLETE INFORMATION: at least one player is uncertain about another player's payoff function.

Setting (continued)

- EXAMPLE: sealed-bid auction.
  - Each bidder knows his own valuation of the good being sold.
  - But does not know any other bidder's valuation.
  - Because bids are submitted in sealed envelopes, players' moves can be thought of as SIMULTANEOUS.

[Dynamic games of incomplete information $\Longrightarrow$ Next chapter]

Objective

- Introduce the concept of Bayesian Nash equilibrium.
- Definitions are a bit abstract and complex.
- So, we first examine an example in detail.

An example: Cournot competition under asymmetric information

- Cournot duopoly model with inverse demand

$$P(Q) = a - Q.$$

  where $Q = q_1 + q_2$.
- FIRM 1's cost function: $C_1(q_1) = cq_1$.
- FIRM 2's cost function:

$$C_2(q_2) = \begin{cases} c_H q_2 & \text{with probability } \theta \\ c_L q_2 & \text{with probability } 1 - \theta \end{cases}$$

  with $c_H > c_L$.

[Firm 2 is a new entrant to the industry or has just invented a new technology].

An example : Information

- Firm 2 knows for sure whether its cost is high or low.
- Firm 1 does not know that for sure. $\Longrightarrow$ASYMMETRIC INFORMATION.
- ASSUMPTION: Firm 1 knows that Firm 2's cost is high with probability θ. Firm 2 knows that Firm 1 knows that and so on and so forth ad infinitum.

An example: Firm 2's choices

- Firm 2 may want to choose different quantities depending on its marginal cost c_H or c_L.
  $\Longrightarrow$ Let $q_2^*(c_H)$ and $q_2^*(c_L)$ denote its choice as a function of its cost.
- IF THE COST IS HIGH and given firm 1's choice q_1^*, $q_2^*(c_H)$ solves

$$\max_{q_2} \left[a - q_1^* - q_2\right] q_2 - c_H q_2 \Longleftrightarrow q_2^*(c_H) = \frac{a - q_1^* - c_H}{2}.$$

- IF THE COST IS LOW and given firm 1's choice q_1^*, $q_2^*(c_L)$ solves

$$\max_{q_2} \left[a - q_1^* - q_2\right] q_2 - c_L q_2 \Longleftrightarrow q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}.$$

An example: Firm 1's choice

- Recall that Firm 1 knows that Firm 2's cost is high with probability θ [and thus low with probability $1-\theta$].
- Given $q_2^*(c_H)$ and $q_2^*(c_H)$, Firm 1's expected payoff is

$$\theta \times \underbrace{\left[a - q_1 - q_2^*(c_H) - c\right] q_1}_{\text{Firm 2's cost is high}} + (1-\theta) \times \underbrace{\left[a - q_1 - q_2^*(c_L) - c\right]}_{\text{Firm 2's cost is low}} q_1 .$$

- Firm 1 chooses q_1 to maximize this expression. FOC yields:

$$q_1 = \frac{\theta\left[a - q_2^*(c_H) - c\right] + (1-\theta)\left[a - q_2^*(c_L) - c\right]}{2}.$$

An example: Solution

- Assuming interior solutions, solve the system of equations with 3 unknown variables, q_1, $q_2^*(c_H)$ and $q_2^*(c_L)$.

$$\begin{cases} q_2^*(c_H) = \frac{a-2c_H-c}{2} + \frac{1-\theta}{6}(c_H - c_L), \\ q_2^*(c_L) = \frac{a-2c_L-c}{2} + \frac{1-\theta}{6}(c_H - c_L), \\ q_1 = \frac{a-2c+\theta c_H+(1-\theta)c_L}{3}. \end{cases}$$

[Compared to the full information case, Firm 2 produces more when its cost is large and less when its cost is low].

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Static Bayesian Games: Types

FIRST STEP: capture the idea that each player knows his own payoffs but may be uncertain about the other players' payoffs.

- Player i may be of different types t_i.
- Each type t_i corresponds to a different payoff function that player i might have.
- Call T_i the type space for Player i, consisting of the collection of all types that Player i might have.

Types (continued)

- EXAMPLE: COURNOT DUOPOLY WITH INCOMPLETE INFORMATION.
  - Two types for Player 2: high cost or low cost. $\Longrightarrow T_2 = \{c_H, c_L\}$.
  - One type for Player 1. $\Longrightarrow T_1 = \{c\}$.

Beliefs

- Any player knows his type $t_i \in T_i$, but may be uncertain about the other players' types

$$t_{-i} \equiv (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) \in T_{-i} \equiv T_1 \times ... \times T_{i-1} \times T_{i+1} \times ... \times T_n.$$

- He thus has a BELIEF about the other players' types t_{-i}, given his knowledge of his own type t_i.
- Belief described by a PROBABILITY DISTRIBUTION $p_i\left(t_{-i} \,|\, t_i\right)$.

Beliefs

Example: Cournot duopoly with incomplete information.

- Player 1's belief about Player 2's type is described by:

$$\begin{aligned} p_1\left(c_H\right) &= \theta, \\ p_1\left(c_L\right) &= 1-\theta. \end{aligned}$$

- As there is only one type for Player 1, Player 2 is sure about what Player 1 is.
  [This can be summarized by the degenerated belief $p_2\left(c\right)=1$]
- Note that each Player's belief is here independent of his own type.

Normal-Form Representation

Definition

The NORMAL-FORM REPRESENTATION of an n-player static Bayesian game specifies:

- the players' action spaces $A_1, ..., A_n$,
- their type spaces $T_1, ..., T_n$,
- their beliefs $p_1, ..., p_n$,
- and their payoff functions $u_1, ..., u_n$.

Normal-Form Representation (continued)

- Player i's type $t_i \in T_i$ is privately known by him [unless there is just one type], at determines his payoff function $u_i\left(a_1, ..., a_n; t_i\right)$.
- Player i's belief $p_i\left(t_{-i}\left|t_i\right.\right)$ describes his uncertainty about the $n-1$ other players' possible types t_{-i} given his own type t_i.
- The game is denoted

$$\mathcal{G} = \left(A_1, ..., A_n; T_1, ..., T_N; p_1, ..., p_n; u_1, ...u_n\right).$$

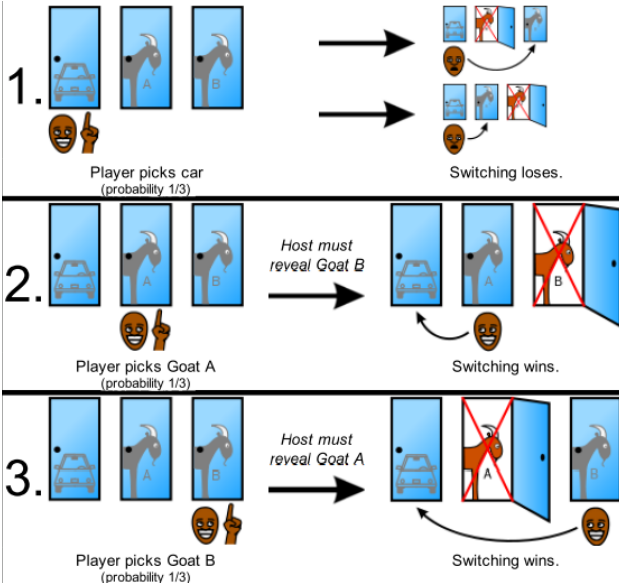
"Timing" of a Static Bayesian Game (Following Harsanyi, 1967)

1. NATURE draws a type t_i for every player in his/her type set T_i.
2. NATURE reveals his/her own type t_i to every player but not to any other player.
3. Players simultaneously choose their actions $a_1, ..., a_n$ in $A_1, ..., A_n$.
4. Payoffs are received. $[u_i\left(a_1, ..., a_n; t_i\right)$ for Player $i]$

An Example: The Monty Hall Problem

- Suppose you're on a game show, and you're given the choice of three doors:
- Behind one door is a car;
- behind the others, goats.
- You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.
- He then says to you, "Do you want to pick door No. 2?"
- Is it to your advantage to switch your choice?

Explanation



Bayesian Players

ASSUMPTION: in STEP 1 of a Bayesian game, it is common knowledge that NATURE draws a type vector $t = (t_1, \ldots, t_n)$ according to the PRIOR probability distribution $p(t)$.

- When Nature reveals t_i to Player i, he/she can compute the belief $p_i(t_{-i} | t_i)$ using Bayes' rule:

$$p_i(t_{-i} | t_i) = \frac{p(t_{-i} \cap t_i)}{p(t_i)} = \frac{p(t_{-i} \cap t_i)}{\sum_{t_{-i} \in T_{-i}} p(t_{-i} \cap t_i)}.$$

- The other players can compute the various beliefs that i might have, depending on i's type, namely $p_i(t_{-i} | t_i)$ for each t_i in T_i.
- In practice, often assumed that $p_i(t_{-i} | t_i)$ does not depend on t_i $\Longrightarrow$ the other players know i's belief about their types [and not only that this belief is $p_i(t_{-i} | t_i)$ if i's type is t_i].

From Incomplete to Imperfect Information

- Thanks to the introduction of NATURE, we have transformed the game of INCOMPLETE information into a game of IMPERFECT information!

INCOMPLETE INFORMATION		IMPERFECT INFORMATION
some players are uncertain about some other players' payoffs.	$\Longrightarrow$	at some move in the game the player with the move does not know the complete history of the game so far.

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Bayesian Nash Equilibrium

- INTUITION: the BNE of a static game of incomplete information is the Nash equilibrium of the corresponding game of imperfect information.
- Since a NE is a strategy profile, what is a STRATEGY in a Bayesian Game?

Strategies

Definition

In the static Bayesian game
$\mathcal{G} = (A_1, ..., A_n; T_1, ..., T_N; p_1, ..., p_n; u_1, ...u_n)$, a strategy for Player i is a function s_i, where for each type t_i in T_i, $s_i(t_i)$ specifies the action from the feasible set A_i that type t_i would choose if drawn by nature.

- A strategy = a complete plan of actions, specifying one action for every possible state of the world, i.e., one action for every possible type.
- Consider the COURNOT DUOPOLY example above. Player 2 has two possible types, c_H and c_L.
  - A strategy for Player 2 specifies a quantity if c_H realizes AS WELL AS a quantity if c_L realizes. $\Longrightarrow$ Here, a strategy = TWO quantities!
  - A strategy for Player 1 consists of one quantity since there is only one type for this player.

Strategies (continued)

- We would not be able to apply the notion of NE to Bayesian games if we allowed a player's strategy not to specify what the player would do if some types were drawn by nature.
- This argument is analogous to the one used for dynamic games of complete information (and SPE).

Bayesian Nash Equilibrium

Definition

In the static Bayesian game
$\mathcal{G} = (A_1, ..., A_n; T_1, ..., T_N; p_1, ..., p_n; u_1, ...u_n)$, the strategies $s^* = (s_1^*, ..., s_n^*)$ are a pure strategy Bayesian Nash equilibrium if, for each player i and for each of i's types t_i in T_i, $s_i(t_i)$ solves

$$\max_{a_i \in A_i} \underbrace{\sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1), ..., s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..., s_n(t_n^*); t_i\right) p_i(t_{-i} | t_i)}_{\text{Player } i\text{'s expected utility from playing } a_i \text{ given his beliefs and the other players' choices}}$$

- In a BNE, no player wants to modify his/her strategy, even if the change involves only one action by one type.
- Definition can be extended to mixed strategies.

Application

- Battle of the sexes in which the husband is not sure of his wife's preferences:

W / H	*Football*	*Opera*
Football	1, 3	0, 0
Opera	0, 0	3, 1

"Loving"

W / H	*Football*	*Opera*
Football	0, 3	3, 0
Opera	3, 0	0, 1

"Leaving"

- This is a game of INCOMPLETE INFORMATION because H does not know W's payoffs.

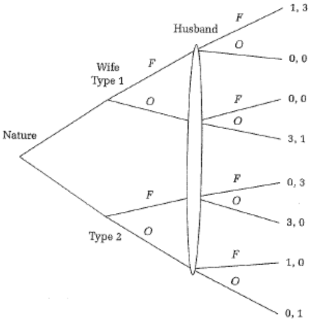
Application: Assumptions

The incomplete information game is transformed into an IMPERFECT INFORMATION game.

- The wife knows her preferences. She has two types "loving" or "leaving.
- The husband does not know his wife's real preferences; he attaches a probability ρ to the fact that she is "loving" and $1-\rho$ to the fact that she is leaving.
- The wife knows her husband's estimates of her preferences, i.e., she knows the value of ρ.

[ASSUMPTION OF COMMON PRIOR: there is a common estimate ρ that both players know. The couple maybe belongs to a population in which ρ% of the wives actually love their husband. Both players know that and – in addition – the wife knows to which group she belongs].

Application: Game Tree



Application: Pure-Strategy BNE

The husband has two options in pure strategies: playing F or playing O.

- Suppose the husband plays F for sure. In a best response,
  - a type 1's wife plays F;
  - a type 2's wife plays O.
- Does the husband maximize his expected payoffs by playing F against (F, O)?
  - By playing F, he gets $\rho \times 3 + (1 - \rho) \times 0 = 3\rho$.
  - By playing O, he gets $\rho \times 0 + (1 - \rho) \times 1 = 1 - \rho$.
  - F is a best response to (F, O) iff $3\rho \geqq 1 - \rho \iff \rho \geqq 1/4$.

Solution

If $\rho \geqq 1/4$, there is a BNE in which he plays F and she plays (F, O).

Application: Pure-Strategy BNE (continued)

- Suppose the husband plays O for sure. In a best response,
  - a type 1's wife plays O;
  - a type 2's wife plays F.
- Does the husband maximize his expected payoffs by playing O against (O, F)?
  - By playing O, he gets $\rho \times 1 + (1 - \rho) \times 0 = \rho$.
  - By playing F, he gets $\rho \times 0 + (1 - \rho) \times 3 = 3(1 - \rho)$.
  - O is a best response to (O, F) iff $\rho \geqq 3(1 - \rho) \iff \rho \geqq 3/4$.

Solution

If $\rho \geqq 3/4$, there is a BNE in which he plays O and she plays (O, F).

Application: Summary

- There are two pure-strategy BNE when $\rho \geqq 3/4$: $((O, F), O)$ and $((F, O), F)$.
- There is only one pure-strategy BNE when $1/4 \leqq \rho \leqq 3/4$: $((F, O), F)$.
- There is no pure-strategy BNE when $\rho < 1/4$. $\Longrightarrow$ Look for mixed-strategy BNE.

Application: Mixed-Strategy BNE

- A mixed strategy for the Husband is a probability distribution over F and O. Call q the probability that he plays F.
- A mixed strategy for the Wife is:
  - a probability for the "loving" wife of paying F: p_1.
  - a probability for the "leaving" wife of paying F: p_2.

Application: Mixed-Strategy BNE (cont'd)

Definition

A mixed strategy BNE of the game is a triple $((p_1, p_2), q)$ in which each player plays a best response, as follows:

- p_1 maximizes a loving wife's payoffs if the husband selects F with probability q and O with probability $1 - q$;
- p_2 maximizes a leaving wife's payoffs if the husband selects F with probability q and O with probability $1 - q$;
- q maximizes the husband's expected payoffs if he believes that his wife is loving with probability ρ (and thus "leaving" with probability $1 - \rho$) [belief].

Application: Mixed-Strategy BNE (cont'd)

Let q be given. Then, what is the wife's best response?

- A loving wife gets q from playing F and $3(1-q)$ from playing O. Plays a mixed strategy only when indifferent between F and $O \iff q = 3/4$.
- A leaving wife gets $3(1-q)$ from playing F and q from playing O. Plays a mixed strategy only when indifferent between F and $O \iff q = 3/4$.

Application: Mixed-Strategy BNE (cont'd)

Now consider the husband's best response to his wife's strategy (p_1, p_2).

- He gets

$$\begin{cases} \rho \times [3p_1 + 0\,(1 - p_1)] + (1-\rho) \times [3p_2 + 0\,(1-p_2)] & \text{if he plays } F, \\ \rho \times [0p_1 + 1 - p_1] + (1-\rho) \times [0p_2 + 1 - p_2] & \text{if he plays } O. \end{cases}$$

- So, he randomizes only if he gets the same payoff from F and O, i.e.,

$$\rho \times [4p_1 - 1] = (1-\rho) \times [1 - 4p_2] \ .$$

Application: Mixed-Strategy BNE (cont'd)

Solution

Given the common prior ρ, mixed-strategy BNE are characterized by $q = 3/4$ and (p_1, p_2) satisfying

$$\rho \times [4p_1 - 1] = (1 - \rho) \times [1 - 4p_2] \ .$$

Example

The equation in (p_1, p_2) is always satisfied, irrespective of the value of ρ, when $4p_1 - 1 = 1 - 4p_2 = 0$, i.e., $p_1 = p_2 = 1/4$. This is an example of mixed-strategy BNE in which the belief does not play a direct part.