5. Game Theory

Static Games of Complete Information

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Setting

Focus on non-cooperative games with the following properties:

- complete information
- Perfect information: players know exactly what happened in previous moves and there are no simultaneous moves.

How will players make their choice?

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Dominant and dominated strategies

- You are a player in a game. You would like to predict the other players' behaviour because what you get depends on what they play.

Example

Think about the "grading game".

- Idea I: A rational player should never play an action for which his/her utility level is less than that obtained for any other possible action, irrespective of what the other players play.
- Idea II: A rational player should always play an action for which his/her utility level is larger than that obtained for all other possible actions, irrespective of what the other players play.

The following definitions capture these basic ideas.

Dominance: definition

Consider the game in strategic form $(X_1, \ldots, X_i, \ldots, X_n; u_1, \ldots, u_i, \ldots, u_n)$, where X_i and u_i represent the strategy set and the payoff function of player i.

Definition (Weak dominance)

Player i's strategy $x_i \in X_i$ *weakly dominates* strategy $x_i' \in X_i$ if and only if

$$u_i\left(x_i, x_{-i}\right) \geq u_i\left(x_i', x_{-i}\right) \text{ for all } x_{-i} \in X_{-i}$$

with a strict inequality for at least one $x_{-i} \in X_{-i}$.

Definition (Strict dominance)

Player i's strategy $x_i \in X_i$ *strictly dominates* strategy $x_i' \in X_i$ if and only if

$$u_i\left(x_i, x_{-i}\right) > u_i\left(x_i', x_{-i}\right) \text{ for all } x_{-i} \in X_{-i}$$

Dominated and dominant strategies: definitions

- A player's strategy is weakly dominated if and only if there exists another strategy which weakly dominates it.
- Player i's strategy x_i strictly dominated if and only if there exists another strategy which strictly dominates it.
- A weakly dominant strategy weakly dominates all other strategies available to a player.
- A strictly dominant strategy strictly dominates all other strategies available to a player.

Dominated/dominant strategies: example

Find the weakly and strictly dominant strategies in the following games.

Player 2

Player 1		c_1	c_2
	r_1	5,2	4,2
	r_2	3,2	3,1

Player 2

Player 1		c_1	c_2
	r_1	2,2	0,2
	r_2	2,0	1,1

Dominant strategy equilibrium: Definition

Definition

A <u>weakly dominant strategy equilibrium</u> is a strategy profile $(x_1, ..., x_N) \in \prod_{n=1}^{N} X_n$ in which every x_n is a weakly dominant strategy for player n, $n = 1, ..., N$.

Definition

A <u>strictly dominant strategy equilibrium</u> is a strategy profile $(x_1, ..., x_N) \in \prod_{n=1}^{N} X_n$ in which every x_n is a strictly dominant strategy for player n, $n = 1, ..., N$.

Example: The prisoner's dilemma

- Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the *same deal*:
  - If one testifies (defects) for the prosecution against the other and the other remains silent (cooperates with the other suspect), the betrayer goes free and the silent accomplice receives the full 10-year sentence.
  - If both remain silent, both prisoners are sentenced to only six months in jail for a minor charge.
  - If each betrays the other, each receives a five-year sentence.
- Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?
- The grading game corresponds to a "prisoner's dilemma".

Dominant strategy equilibrium in the Prisoner's dilemma

Find the dominated strategies. Which should be the outcome of the game?

$1\backslash 2$	Silent	Betray
Silent	$-1/2, -1/2$	$-10, 0$
Betray	$0, -10$	$-5, -5$

Dominant strategy equilibrium: example

We change the payoffs of the prisoner's dilemma as follows:

1\2	Silent	Betray
Silent	$-1, -1$	$0, 0$
Betray	$0, 0$	$-5, -5$

Which should be the outcome of the game?

- Here no equilibrium when using dominance principles.
- Other principles are needed to predict the outcome of the game.

Rule #1: Never play a dominated strategy!
[A rational player applies Rule #1 and you know that he/she does so and he/she knows that you know that he/she does so, etc.]

Rule #2: Always play your dominant strategy!
[A rational player applies Rule #2 and you know that he/she does so and he/she knows that you know that he/she does so, etc.]

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Iterated Dominance: example

Idea: rational players should recognize that other players have dominated strategies, and should act accordingly.

- Consider the 2-player simultaneous-move game described by the following bimatrix:

1\2	D	E	F	G
A	3, 3	1, 0	0, 3	2, 2
B	3, 3	0, 0	3, 2	0, 2
C	4, 1	2, 2	2, 0	3, 1

- For player 1, C strictly dominates A. So, A will never be played by A .

Iterated Dominance: example (continued)

- As Player A will never play A.
- Because there is complete and perfect information, player 2 knows it too.
- We can just delete it in the payoff matrix!

1\2	D	E	F	G
B	3, 3	0, 0	3, 2	0, 2
C	4, 1	2, 2	2, 0	3, 1

Iterated Dominance: example (continued)

- Then F is strictly dominated by D and can be deleted:

1\2	D	E	G
B	3, 3	0, 0	0, 2
C	4, 1	2, 2	3, 1

Iterated Dominance: example (continued)

- B is dominated by C. So:

1\2	D	E	G
C	4, 1	2, 2	3, 1

- Finally, D and G are dominated by E and can thus be deleted:

1\2	E
C	2, 2

- There is only one option left: (C, E).

Iterated dominance equilibrium: definition

Definition

The game is dominance solvable if the process of iterated deletion of strictly dominated strategies leads to strategy profiles between which the players are indifferent. These profiles are called iterated (strict) dominance equilibrium.

- In the above example, the strategy profile (C, E) is an iterated (strict) dominance equilibrium.
- Are there iterated dominance equilibria if one considers the 2-player simultaneous-move game described by the following bimatrix?

1\2	C	D	E
A	3, 3	0, 0	3, 2
B	4, 1	2, 2	2, 2

Iterated dominance equilibrium: Another example

1\2	D	E
A	0, 0	2, 5
B	5, 5	100, 5
C	5, 5	0, 0

- Apply iterated dominance, applied to weakly dominated strategies.
- Conclusion?

Iterated dominance equilibrium: remarks

- If iterated dominance is used on weakly dominated strategies, the order of elimination may matter.
- If iterated dominance is used only on strictly dominated strategies the order does not matter, and we can remove all strictly dominated strategies at once.
- Method best suited for strict dominance. So, **we will only apply it to strictly dominated strategies**.

Rule #3: Delete all strictly dominated strategies sequentially
[A rational player applies Rule #3 and you know that he/she does so and he/she knows that you know that he/she does so, etc.]

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Nash equilibrium

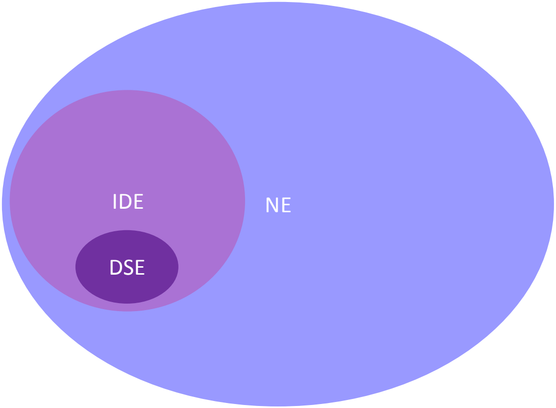
Definition (Nash Equilibrium)

A strategy profile $(x_1^*, \ldots, x_n^*)$ of the game $(X_1, \ldots, X_n)$ is a Nash Equilibrium (NE) if and only if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \text{ for all } x_i \in X_i \text{ and for all } i. \tag{1}$$

- $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$.
- The strategy chosen by any player is a best response to the strategies chosen by all other players. Think about the best responses derived for Matching Pennies in the last section of Chapter 1. We found a Nash equilibrium, which was supported by mixed strategies.
- STABILITY: best response to best responses $\Longrightarrow$ none of the players has an incentive to unilaterally deviate from his/her choice.

NE compared to IDE and DSE



Best response correspondences

What is a CORRESPONDENCE?

- Generalizes the concept of function;
- A correspondence f between two sets A and B is a mapping $f : A \longrightarrow P(B)$ from the elements of the set A to the power set of B (i.e., to the set of all subsets of B).
- "A mapping in which the image set is a set (and not just a single number)".
- Often denoted $f : A \longrightarrow\longrightarrow B$ or $f : A \rightrightarrows B$.

Best response correspondences (continued)

For any $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i}$, player i's BEST RESPONSE (BR) is the correspondence $B_i : \Delta X_{-i} = \prod_{\substack{j=1,\ldots,n \\ j \neq i}} \Delta X_j \longrightarrow\!\!\!\!\longrightarrow \Delta X_i$, with

$$B_i(x_{-i}) = \{x_i \in \Delta X_i : u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i}) \text{ for all } x_i' \in \Delta X_i\}.$$

- Assume you are i and all other players have chosen their strategies.
  $\Longrightarrow x_{-i}$ given.
- Then, what should you choose to maximize your utility?
  $\Longrightarrow$ choose any element of $B_i\left(x_{-i}\right)$.

Alternative definition of NE

Definition

A strategy profile $(x_1^*, \dots, x_n^*)$ of the game $(X_1, \dots, X_n)$ is a Nash Equilibrium if and only if

$$x_i^* \in B_i(x_{-i}^*) \text{ for all } i.$$

- In a NE, every player's strategy is a best response to all other players' strategies.
- Provide us with a method for finding NE of a game:

1. Derive BR for each player;
2. Find the strategy profile that satisfies the definition of a NE.

Using BR to find NE

METHODOLOGY:

1. Derive every player's BR, i.e., find

$$B_i(x^{-i*}) \in \arg\max_{x_i \in \Delta X_i} u_i(x_i, x_{-i}) \text{ for every } x_{-i} \in \Delta X_{-i}.$$

2. Find the strategy profile $(x_1^*, \ldots, x_n^*)$ that satisfies the definition for a NE, i.e.,

$$x_i^* \in B_i(x^{-i*}) \text{ for all } i. \tag{2}$$

Example 1

- PRISONER'S DILEMMA. Find the NE of the following 2-player simultaneous-move game:

1\2	Silent	Betray
Silent	$-\frac{1}{2}, -\frac{1}{2}$	$-10, 0$
Betray	$0, -10$	$-5, -5$

Example 2

- BATTLE OF THE SEXES. Imagine a couple that agreed to meet this evening, but cannot recall if they will be attending the opera or a football match.
  - The husband would most of all like to go to the football game.
  - The wife would like to go to the opera.
  - Both would prefer to go to the same place rather than different ones.
- If they cannot communicate, where should they go?

1\2	Opera	Football
Opera	3, 2	0, 0
Football	0, 0	2, 3

Example 3

- MATCHING PENNIES. Find the NE of the following 2-player simultaneous-move game:

$1\backslash 2$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Existence

A game is finite if it has finitely many players, each of whom has a finite set of strategies (i.e., a finite number of strategies).
A compact set is closed and bounded.

Theorem

Every finite game posseses at least one NE, possibly in mixed strategies.

Theorem

A game $(X_1, \ldots, X_n; u_1, \ldots, u_n)$ *has at least one NE, possibly in mixed strategies, if for every player* i*:*

- *the strategy set* X_i *is a non-empty compact subset of* $\mathbb{R}^n$;
- *the payoff function* u_i *is continuous.*

Justification of NE

- Pre-play negociations.
- Focal-point principle and social convention.
- Coordination by exterior entity.
- Habits.