Chapter 3
Introduction to the General Equilibrium and to Welfare Economics

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Roadmap

- Introduction
- Pareto Optimality
- General Equilibrium
- The Two Fundamental Theorems of Welfare Economics
- Taxation and individuals

Objectives of this Chapter

- Make sure that we all agree on the basic general equilibrium, i.e., on its definition and its welfare properties.
  - What is meant by an "efficient" economic outcome.
  - Under which conditions the process through which prices are determined leads to an efficient outcome.
  - Whether efficiency and equity are compatible or if there is a trade-off between more efficiency and more equity.
- Refresh and/or introduce a number of basic results and concepts from (public) economics.

Taking All Market Interations into Account

Consider a setting in which:

- Prices are parameters for the individual decision-takers (consumers and producers).
- Hence, demand and supply functions depend on:
  - the price of the commodity we consider;
  - but also on the other prices in the economy.
- How prices are determined by the interaction of the decisions of individuals?
  - Focus on the own price effect $\Rightarrow$ partial equilibrium viewpoint: interactions between the markets are neglected; single marker considered in isolation.
  - Taking all interactions into account $\Rightarrow$ general equilibrium viewpoint.

Competitive General Equilibrium

Our analysis will sometimes focus on the general equilibrium under the assumptions of perfect competition:

- Many buyers and many sellers. Economic agents are thus "price takers".
- Homogeneous products. All products exchanged on a given market, have comparable features.
- No-entry/exit barriers. It is easy to enter or exit as a business in a perfectly competitive market. Hence, if there is a market on which firms make strictly positive profits in the short-run, new firms will enter the market. New firms will continue to enter the market up to the point where profits are equal to zero. Consequently, in the long run, firms do not make positive profits.
- Perfect information. All information relevant to the consumer's/producer's decision is available at no cost.

Why Is Perfect Competition Interesting?

- All distinguishing characteristics of competitive markets are (usually) not satisfied at the same time. $\Rightarrow$ General equilibrium under imperfect competition:
  - market powers;
  - asymmetric information;
  - externalities;
  - public goods.
- However perfect competition is worth studying because:
  - benchmark to which the equilibrium under imperfect competition can be compared;
  - attractive features. Might (to some extent) be regarded as a situation we would like to achieve.

Roadmap

Pareto Criterion

Definition

Consider two allocations a_1 and a_2 of economic resources. Then, the allocation a_1 is Pareto-preferred to the allocation a_2 if and only if no one is worse off under a_1 than a_2.

- Consider an economy with two individuals with the same preferences $u(x_1, x_2) = x_1 x_2$.
- Now, consider the allocations $\left(x_1^A, x_2^A, x_1^B, x_2^B\right)$ defined by $X^1 = (1, 1, 2, 2)$, $X^2 = (1, 2, 1, 2)$ and $X^3 = (1.5, 1.5, 2, 2)$.
- What can we say according to the Pareto criterion? X_3 Pareto-preferred to X_2. Incomplete ranking.

Utility	A	B
X^1	1	4
X^2	2	2
X^3	2.25	4

Pareto Optimality/Efficiency

Definition

A Pareto efficient allocation is an allocation of economic resources which is Pareto-preferred to all other feasible allocations.

This implies that:

- There is no way to make all agents better off.
- In other words, it is not possible to increase the welfare of an individual without decreasing that of another individual. item In the above example, X^3 is the unique Pareto optimal allocation (in the set consisting of X^1, X^2 and X^3).

Pure Exchange Economy: Basic Setting

- All economic agents are consumers.
- 2 agents (A and B) + 2 goods (1 and 2) + utilities $u^A\left(x_1^A, x_2^A\right)$ and $u^B\left(x_1^B, x_2^B\right)$.
- Quantities consumed: $\left(x_1^A, x_2^A\right)$ for A and $\left(x_1^B, x_2^B\right)$ for B.
- Initial endowments: A has $\left(\omega_1^A, \omega_2^A\right)$ of goods 1 and 2 respectively; B has $\left(\omega_1^B, \omega_2^B\right)$.
- Economy represented in an Edgeworth box of size $X_1 \times X_2 = \left(\omega_1^A + \omega_1^B\right) \times \left(\omega_2^A + \omega_2^B\right)$.

Characterization of Pareto Optimum Allocations (1.)

What are the allocations of resources corresponding to Pareto optima? Solution to:

Problem

Find a 4-uplet $\left(x_1^A, x_2^A, x_1^B, x_2^B\right)$ *which maximises* $u^A\left(x_1^A, x_2^A\right)$ *subject to the constraint that person* B*'s utility is above a threshold* $\overline{u}_B$, *i.e.*

$$u^B\left(x_1^B, x_2^B\right) \geq \overline{u}_B,$$

and to the resource constraints

$$x_1^A + x_1^B \leq \omega_1^A + \omega_1^B = X_1,$$
$$x_2^A + x_2^B \leq \omega_2^A + \omega_2^B = X_2.$$

Underlying assumptions (1.)

We will asssume that the utility functions are of class C^2, increasing in each argument and quasiconcave.
Let $\succsim_i$ be person i's preference relations, defined for the choice set X. We define:

- completeness: $\forall (x, y) \in X^2$, $x \succsim_i y$ or $y \succsim_i x$.
- transitivity: $\forall (x, y, z) \in X^3$, $x \succsim_i y$ and $y \succsim_i z$ implies $x \succsim_i z$.
- continuity: if $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$ and $x_n \succsim_i y_n$ for every n, then $x \succsim_i y$.
- monotonicity: $\forall (x, y) \in X^2$, $x > y \Longrightarrow x \succsim_i y$.
- convexity: $\forall (x, y) \in X^2$, $\{y \in X : y \succsim_i x\}$ is convex.

Underlying assumptions (2.)

Under Completeness + Transitivity + Continuity

There exists a continuous utility function $u : X \longrightarrow \mathbb{R}$ which represents person i's preferences.

Under Completeness + Transitivity + Continuity + Monotonicity + Convexity

There exists a utility function $u : X \longrightarrow \mathbb{R}$ of class C^2, increasing in each of its argument and quasiconcave which represents person i's preferences.

Characterization of Pareto Optimum Allocations (2.)

- Under these assumptions, the individual utility maximization problem has (at least) a solution.
  Strict convexity of preference relation (i.e., strict quasiconcavity of utility function) ensures uniqueness of the solution
- Individual budget constraints are binding, implying that the overall resource constraints are binding.
- Monotonicity of person A's preferences implies that the utility constraint $u^B\left(x_1^B, x_2^B\right) \geq \overline{u}_B$ is binding.

Characterization of Pareto Optimum Allocations (3.)

Under the above-mentionned assumptions, looking for a PO allocation is equivalent to solving:

Problem (Equivalent Formulation for PO)

Find a 4-uplet $\left(x_1^A, x_2^A, x_1^B, x_2^B\right) = \left(x_1^A, x_2^A, X_1 - x_1^A, X_2 - x_2^A\right)$ *which maximises* $u^A\left(x_1^A, x_2^A\right)$ *subject to the constraint*

$$u^B\left(X_1 - x_1^A, X_2 - x_2^A\right) = \overline{u}_B.$$

- For any allocation solution to this Problem, it is impossible to increase the utility of person A without decreasing the utility of person B below $\overline{u}_B$.
- When we consider all possible values for the threshold $\overline{u}_B$, from 0 – case in which all goods are given to person A – to $u_B\left(X_1, X_2\right)$ – case in which all goods are given to person B –, we describe all Pareto efficient allocation in the Edgeworth box.

Characterization of Pareto Optimum Allocations (4.)

- Lagrangian:
  $L = u^A\left(x_1^A, x_2^A\right) + \lambda\left[u^B\left(X_1 - x_1^A, X_2 - x_2^A\right) - \overline{u}_B\right].$
- First-order conditions for a maximum:

$$\frac{\partial L}{\partial x_1^A} = 0 \Leftrightarrow \frac{\partial u^A\left(x_1^A, x_2^A\right)}{\partial x_1^A} - \lambda \frac{\partial u^B\left(X_1 - x_1^A, X_2 - x_2^A\right)}{\partial x_1^B} = 0, \quad (1)$$

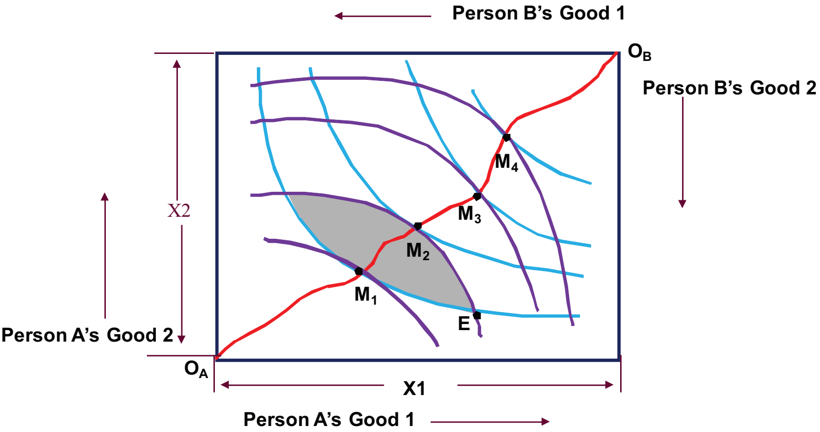
$$\frac{\partial L}{\partial x_2^A} = 0 \Leftrightarrow \frac{\partial u^A\left(x_1^A, x_2^A\right)}{\partial x_2^A} - \lambda \frac{\partial u^B\left(X_1 - x_1^A, X_2 - x_2^A\right)}{\partial x_2^A} = 0. \quad (2)$$

- Eliminating λ, one gets:

$$\frac{\frac{\partial u^A\left(x_1^A, x_2^A\right)}{\partial x_2^A}}{\frac{\partial u^A\left(x_1^A, x_2^A\right)}{\partial x_1^A}} = \frac{\frac{\partial u^B\left(x_1^B, x_2^B\right)}{\partial x_2^B}}{\frac{\partial u^B\left(x_1^B, x_2^B\right)}{\partial x_1^B}} \Leftrightarrow MRS_{12}^A\left(x_1^A, x_2^A\right) = MRS_{12}^B\left(x_1^B, x_2^B\right)$$

- At interior Pareto optimal allocations, the marginal rates of substitution of all individuals are equal.

Pareto Set, Efficiency and Social Justice?



Pareto Optimum with Production

- For simplicity, we will first look at a Robinson Crusoe economy:
  - a consumer (representative consumer) (who thus owns the firm)
  - two goods (labelled 1 and 2) produced in quantities x_1 and x_2 using the limited amount of resources available in the economy.
- Production Possibility Set (PPS): all combinations (x_1, x_2) which can be produced given available resources.
- Production Possibility Frontier (PPF): all technically efficient allocations.

PPS and PPF

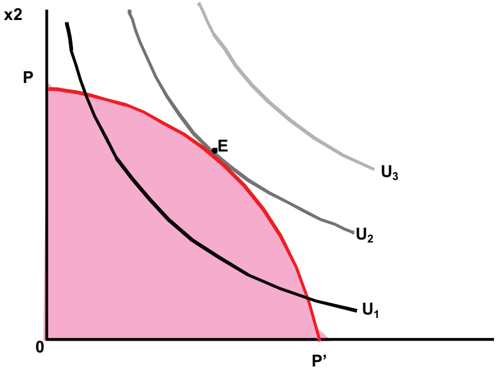


Figure: Production Possibility Set (Pink Area including the Red Frontier)

Pareto Optimum Allocation with Production

- Marginal Rate of Transformation: The (opposite of the) slope of the PPF at a given (x_1, x_2)-combination is called marginal rate of transformation of good 1 for good 2.
- Given individual preferences, best possible outcome at E, which is a Pareto optimum. There, the MRT is equal to the MRS.

By extension,

Theorem

In a Pareto optimum allocation with production:
(i) the marginal rates of substitution in consumption are identical for all consumers;
(ii) the marginal rate of transformation in production is identical for all products;
(iii) The marginal rates of substitution in consumption are equal to the marginal rates of transformation in production, such that production processes match consumer wants.

Roadmap

General Equilibrium

Definition

A general equilibrium is an equilibrium with the following features:

1. every consumer maximises his utility in his budget set;
2. every firm maximize its profits;
3. demand equals supply on every market.

Condition (3.) is called the market clearing condition.

Competitive Equilibrium and Pareto Efficiency: Example

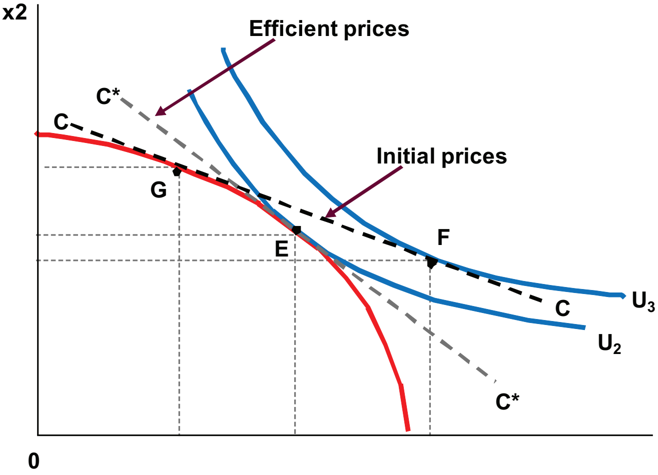


Figure: Equilibrium and Efficiency: An Example

Roadmap

Pareto Efficiency of Any Competitive General Equilibrium

First Fundamental Theorem of Welfare Economics

Any general competitive equilibrium is Pareto efficient.

- Competitive markets tend toward the efficient allocation of resources. Supports a case for non-intervention *in ideal conditions and in ideal conditions only*: let the markets do the work and the outcome will be Pareto efficient.
- Pareto efficiency is not necessarily the same thing as desirability. There can be many possible Pareto efficient allocations of resources and not all of them may be equally "desirable" by society.
- The conditions of perfect competition are often not satisfied. Because the competitive equilibrium appears as normatively attractive (because it is a Pareto optimum), should governments' policies aim at implementing the conditions of perfect competition when they are not satisfied in practice?

Decentralization of a Pareto Optimal Allocation as a General Equilibrium

Any general competitive equilibrium is Pareto efficient. Yet, there can be many possible Pareto efficient allocations of resources and not all of them may be equally "desirable" by society. Is it possible for the policymaker to choose a desirable allocation, implement an appropriate economic policy, then let the markets freely work and finally obtain the desired allocation? The answer is: yes, under certain conditions.

Decentralization of a Pareto Optimal Allocation: Illustration

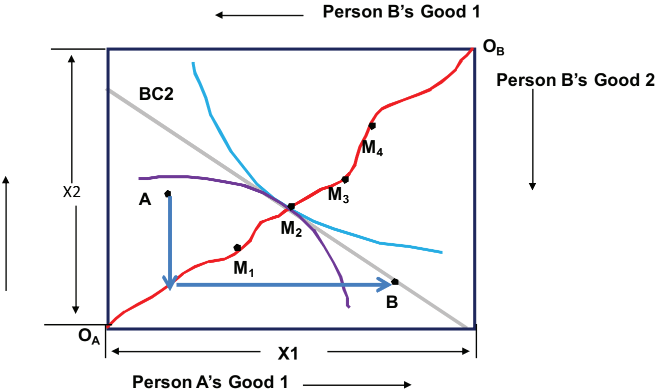


Figure: Decentralization of An Allocation as a General Equilibrium

Decentralization of a Pareto Optimal Allocation

Second Fundamental Theorem of Welfare Economics

When individual preferences and production possibility sets are convex, any Pareto optimum allocation can be obtained as a general equilibrium once appropriate lump-sum transfers have taken place.

- Choice of the socially desirable allocation? $\Rightarrow$ Use of a Paretian social welfare function. Assume individual utilities are $u\left(x_1^i, x_2^i\right)$ for $i = 1, ..., N$. We can define a social welfare function as

$$W\left(u\left(x_1^1, x_2^2\right), ..., u\left(x_1^N, x_2^N\right)\right),$$

  non-decreasing in every individual utility $u\left(x_1^i, x_2^i\right)$. Examples of social welfare functions:

  - Pure Utilitarianism: $W\left(u\left(x_1^1, x_2^2\right), ..., u\left(x_1^N, x_2^N\right)\right) = \sum_{i=1}^N u\left(x_1^i, x_2^i\right)$.
  - Weighted Utilitarianism: $W\left(u\left(x_1^1, x_2^2\right), ..., u\left(x_1^N, x_2^N\right)\right) = \sum_{i=1}^N G\left(u\left(x_1^i, x_2^i\right)\right)$ with G concave (aversion to inequality).
  - Rawlsian Maximin: $W\left(u\left(x_1^1, x_2^2\right), ..., u\left(x_1^N, x_2^N\right)\right) = \min_{i=1,...,N}\left\{u\left(x_1^i, x_2^i\right)\right\}$.

- Are lump-sum transfers feasible?

Roadmap

Taxation and individuals

Throughout the Lectures, we derive key results in public economics by using highly simplified mathematical models of the economy

Generally, we will focus on the *simplest possible* models that still bring home the key message

Some key characteristics of these economic models include:

- Firms are fully rational and maximize profits
- Individuals are fully rational and maximize utility
- Markets are perfectly competitive

Under these assumptions, we derive the following results below:

1. Equivalence between employee's and employer's income taxes
2. Equivalence between income tax and uniform commodity tax
3. Utility equivalence between different tax instruments
4. Income and substitution effects of the income tax

Equivalence between employee's and employer's income taxes

Assume that the income tax consists of an employee's part t and an employer's part τ

Firms hire l units of labor, with constant productivity n per unit
$\Longrightarrow$ total production equals nl

Firms pay wages w and face a payroll tax rate equal to τ
$\Longrightarrow$ total labor costs equal $(1+\tau)wl$

Profits are thus given by:

$$\Pi = nl - (1+\tau)wl$$

Perfect competition and free entry of firms imply that equilibrium wage rate is given by $(1+\tau)w = n$.

A representative individual supplies l units of labor for a gross wage rate of w per unit

$\implies$ earns before-tax income $wl = \left(\frac{1}{1+\tau}\right) nl$

Individual pays income tax rate t and consumes after-tax income c

$\implies$ consumption equal to $c = (1-t)wl = \left(\frac{1-t}{1+\tau}\right) nl$

Individual utility is C^2, increasing in consumption and decreasing in labor supply: $U = u(c, l)$, with $u_c > 0,\ u_l < 0$. Substitute for consumption c:

$$U = u\left(\left(\frac{1-t}{1+\tau}\right) nl, l\right)$$

Maximize with respect to labor supply l:

$$-u_l = \left(\frac{1-t}{1+\tau}\right) nu_c$$

Marginal costs of supplying labor equals marginal benefits of consuming the resulting income

Notice that both utility and labor supply only depends on the combination $\frac{1-t}{1+\tau}$, not on t and τ separately

$\Longrightarrow$ It does not matter whether income taxes are levied through employer taxes ($\tau > 0$, $t = 0$) or employee taxes ($t > 0$, $\tau = 0$) or any other combination.

$\Longrightarrow$ Statements such as "We should shift the burden of income taxation from employees to employers" are largely nonsense.

(Can you think of instances in which this equivalence does not hold?)

In the remainder of the Lectures, we typically assume $\tau = 0$ and thus $w = n$ without loss of generality

Equivalence between an income tax and a uniform commodity tax

We assume the government levies a proportional income tax t and a uniform commodity tax τ

Representative individual with wage w supplies l units of labor:

- Before-tax income given by wl
- After-tax income given by $(1-t)wl$

Income is spent on consumption goods c and x, and on a uniform commodity tax $\tau(c+x)$

Utility is given by:

$$U = u(c,x,l), \qquad u_c, u_x > 0,\ u_l < 0$$

Budget constraint is given by:

$$(1+\tau)(c+x) = (1-t)wl$$

We can rewrite the budget constraint as:

$$c = \left(\frac{1-t}{1+\tau}\right) wl - x$$

Substitute into the utility function to get:

$$U = u\left(\left(\frac{1-t}{1+\tau}\right) wl - x, x, l\right)$$

Maximize utility with respect to x and l:

$$\begin{aligned} u_x &= u_c \\ -u_l &= \left(\frac{1-t}{1+\tau}\right) w u_c \end{aligned}$$

First FOC: no marginal benefits of consuming more x by reducing consumption of c

Second FOC: marginal costs of supplying more l equals marginal benefits of consuming the resulting income

Notice that both utility and labor supply only depends on the combination $\frac{1-t}{1+\tau}$, not on t and τ separately

$\Longrightarrow$ It does not matter whether taxes are levied through a tax on income ($t > 0$, $\tau = 0$) or through a uniform tax on consumption ($\tau > 0$, $t = 0$)

$\Longrightarrow$ Statements such as "We should shift the burden of taxation from income to consumption" are largely nonsense

(Can you think of instances in which this equivalence does not hold?)

In the remainder of the Lectures, we typically assume either $\tau = 0$ or $t = 0$ without loss of generality

Utility equivalence of different tax instruments

We assume the government levies a proportional income tax t, a *specific* commodity tax τ, and a lump-sum tax T

Q: Does it matter for utility which tax instrument is used to raise one unit of taxes?

Representative individual with wage w supplies l units of labor:

- Before-tax income given by wl
- After-tax income given by $(1-t)wl - T$

After-tax income is spent on consumption goods c and x, and on a specific commodity tax τx

Utility is given by:

$$U = u(c, x, l), \qquad u_c, u_x > 0,\ u_l < 0$$

Budget constraint is given by:

$$c = (1-t)wl - T - (1+\tau)x$$

Substitute budget constraint into the utility function to get:

$$U = u((1-t)wl - T - (1+\tau)x, x, l)$$

Still a function of l and x. We want to write utility as a function of tax instruments $\{t, \tau, T\}$ to determine their effects on utility.

First-order conditions w.r.t. x and l are again given by:

$$\begin{aligned} u_x &= u_c \\ -u_l &= \left(\frac{1-t}{1+\tau}\right) w u_c \end{aligned}$$

These conditions imply equilibrium x and l as function of tax instruments: $x^* = x(t, \tau, T)$ and $l^* = l(t, \tau, T)$

Substitute back into utility function to get **indirect utility** – the utility gained after the individual optimizes his behavior:

$$v(t, \tau, T) \equiv u((1-t)wl^* - T - (1+\tau)x^*, x^*, l^*)$$

The envelope theorem: because indirect utility is maximized w.r.t. x and l, partial derivatives w.r.t. x^* and l^* are zero

Taking derivatives w.r.t. $\{T, t, \tau\}$ yields:

$$v_T \cdot \mathrm{d}T = -u_c \cdot \mathrm{d}T$$
$$v_t \cdot \mathrm{d}t = -u_c \cdot wl^* \mathrm{d}t$$
$$v_\tau \cdot \mathrm{d}\tau = -u_c \cdot x^* \mathrm{d}\tau$$

$\Longrightarrow$ The utility effects of raising the lump-sum tax by $\mathrm{d}T$, the income tax by $wl^*\mathrm{d}t$, or the commodity tax by $x^*\mathrm{d}\tau$ are **identical**

$\Longrightarrow$ In other words, **individuals do not care through which tax instruments an additional unit of taxes is collected!** They simply suffer, to an extent equal to their income loss times their marginal utility of consumption u_c

This result is sometimes referred to as **Roy's Identity**

Do at home: derive x^*, l^*, and $v(t, \tau, T)$ for the specific utility function $u(c, x, l) = c + \ln x - \left(\frac{l^{1+1/e}}{1+1/e}\right)$. Confirm the utility equivalence of the different tax instruments by taking partial derivatives of the indirect utility w.r.t. $\{t, \tau, T\}$.

Income and substitution effects of the income tax on labor supply

While utility effects of different tax instruments are identical, **the behavioral effects are not**

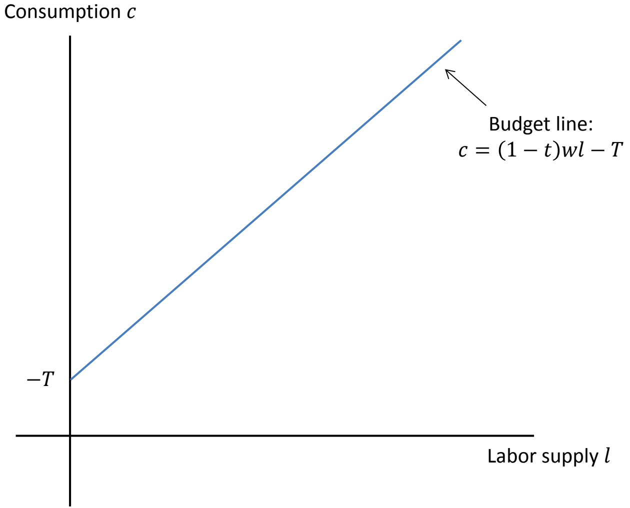
Indeed, the lump-sum tax T only has an **income effect** on labor supply; the income tax rate t has both an income effect and **a substitution effect**

Income effect: intuitively, **both T and t lower disposable income**, thereby decreasing the consumption of commodities and leisure (higher labor supply)

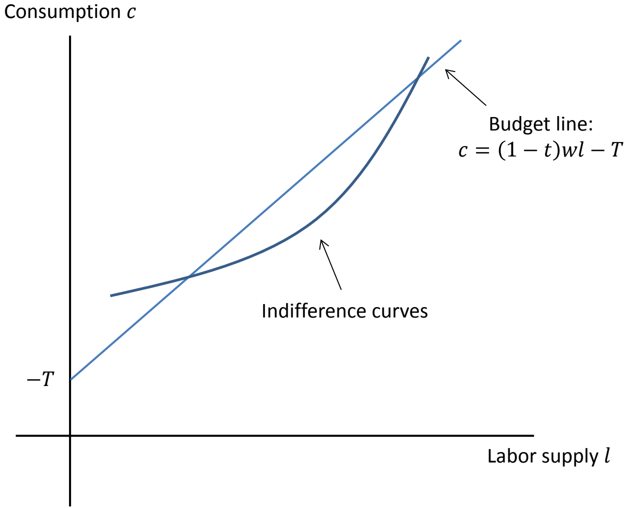
Substitution effect: **the income tax t also raises the price of commodities in terms of leisure**, leading to a substitution effect from commodities to leisure (lower labor supply) [if"normal goods"]

$\Longrightarrow$ the income effect of t on l is positive; the substitution effect is negative; the net effect is ambiguous

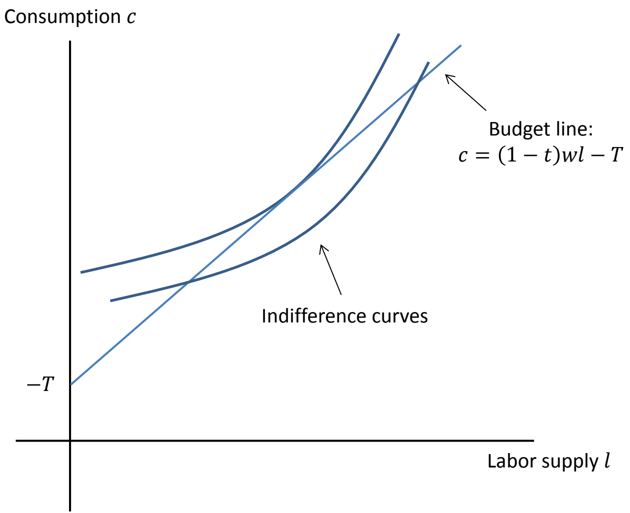
Labor supply: income and substitution effects



Labor supply: income and substitution effects



Labor supply: income and substitution effects



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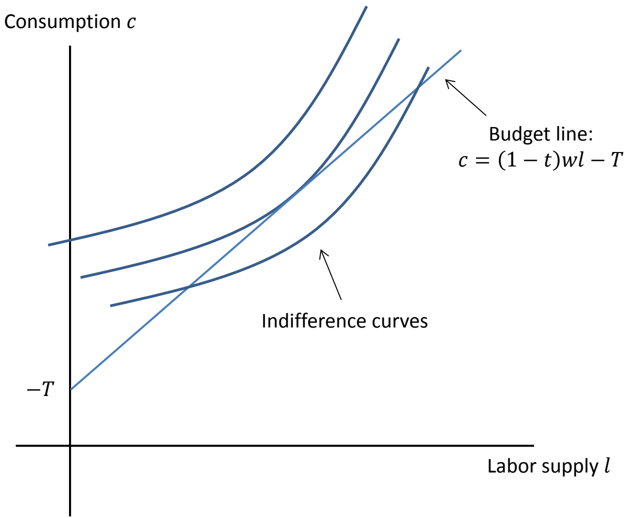
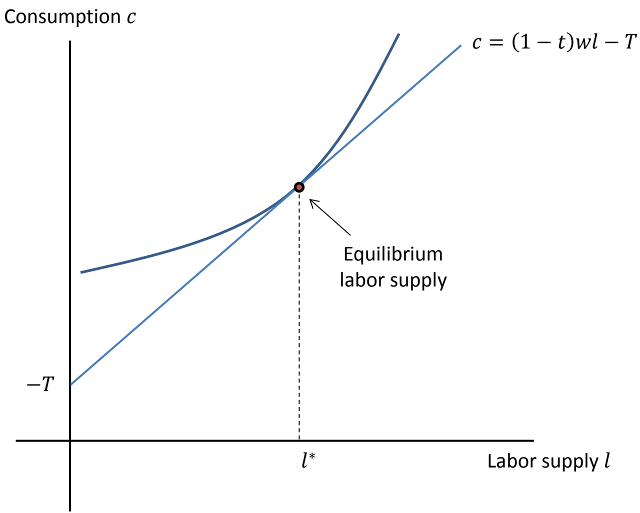
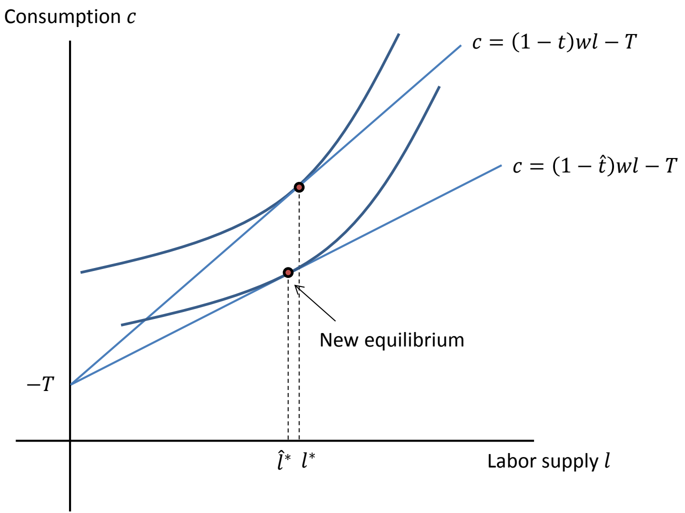


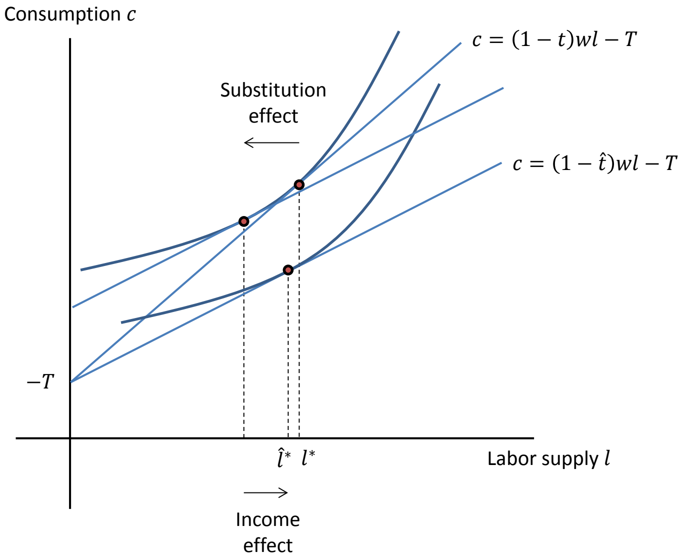
Labor supply: income and substitution effects



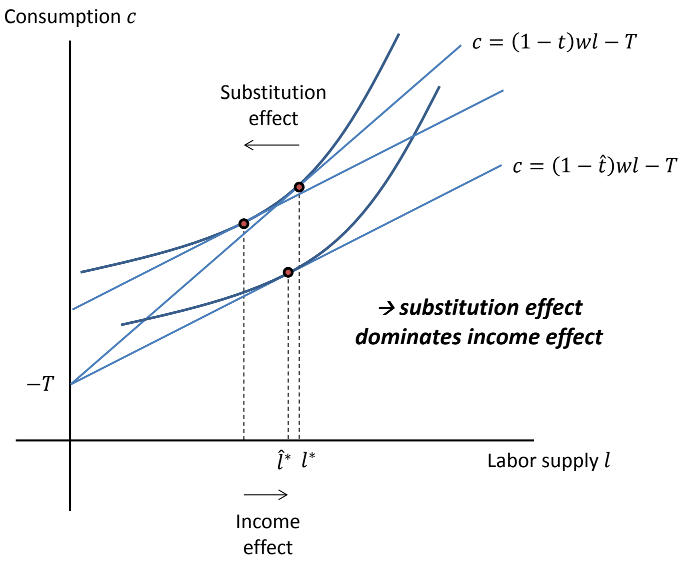
Labor supply: income and substitution effects



Labor supply: income and substitution effects

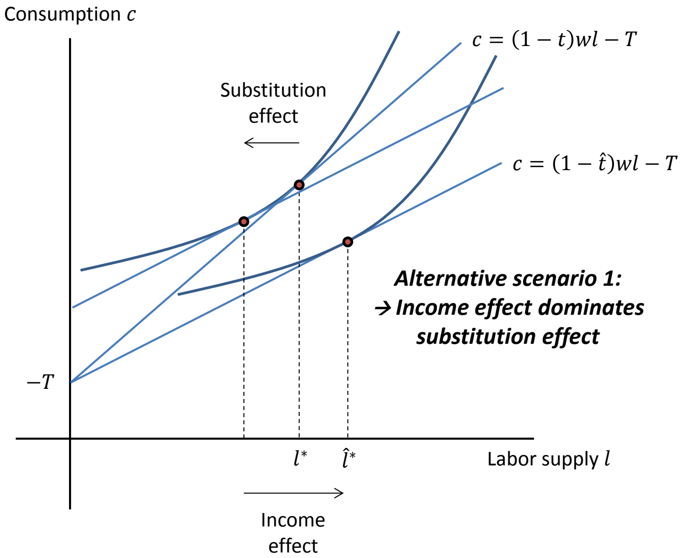


Labor supply: income and substitution effects



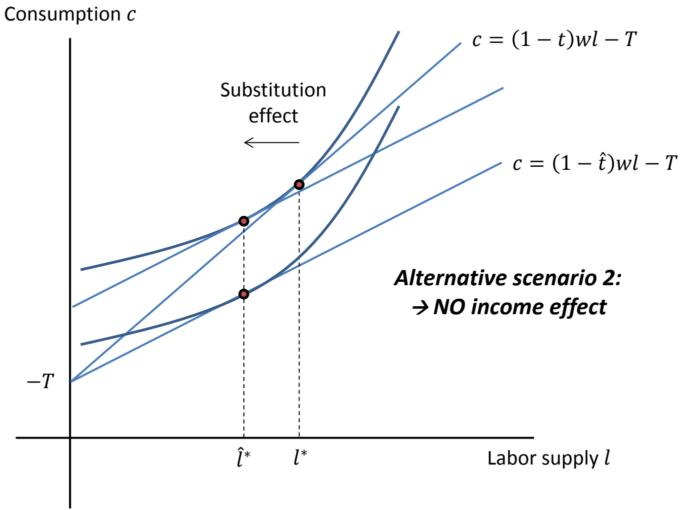
→ substitution effect dominates income effect

Labor supply: income and substitution effects



Alternative scenario 1:
→ Income effect dominates substitution effect

Labor supply: income and substitution effects



Alternative scenario 2:
→ NO income effect

Formally, we can use the **Slutsky decomposition** to decompose the total labor supply response to a change in income taxes $\frac{\mathrm{d}l}{\mathrm{d}t}$ into a substitution effect and an income effect:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\mathrm{d}l^c}{\mathrm{d}t} + wl\frac{\mathrm{d}l}{\mathrm{d}T}$$

$\frac{\mathrm{d}l^c}{\mathrm{d}t}$ is the **compensated change** in labor supply: it represents the change in labor supply due to the change in the slope of the budget line

$wl\frac{\mathrm{d}l}{\mathrm{d}T}$ is the **income effect** on labor supply: it represents the change in labor supply due to the shift of the budget line

Note that only the substitution effect represents the labor supply effect of distorting prices!

Q: We have shown that income taxes and lump-sum taxes **differ in their labor supply effects**. But we have also shown that income taxes and lump-sum taxes have **equivalent utility effects**. How can we explain this seeming paradox?

The net-of-tax rate elasticity of labor supply / taxable income

As the course unfolds, we will often make use of the concept of net-of-tax rate elasticities. For example, the **compensated net-of-tax rate elasticity of labor supply** can be written as:

$$e \equiv \frac{\mathrm{d}l^c}{\mathrm{d}(1-t)} \frac{1-t}{l} = -\frac{\mathrm{d}l^c}{\mathrm{d}t} \frac{1-t}{l}$$

If the net-of-tax rate $(1-t)$ goes up by 1%, labor supply goes up by $e\%$

Instead of labor supply l, we will also often talk about **taxable income** $z \equiv wl$. Notice that the **elasticity of taxable income** is equal to the elasticity of labor supply:

$$\frac{\mathrm{d}z}{\mathrm{d}(1-t)} \frac{1-t}{z} = \frac{w\mathrm{d}l}{\mathrm{d}(1-t)} \frac{1-t}{wl} = \frac{\mathrm{d}l}{\mathrm{d}(1-t)} \frac{1-t}{l}$$