Microeconomics
Chapter 1 - Utility and Choice: the Theory of the Consumer

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- Characterization Using Calculus

Introduction

- Central assumption in the theory of the consumer: optimisation
- "Given the feasible set of consumption bundles, the bundle chosen is the one the consumer prefers"
- A three-steps analysis:
  - How to model the consumer's preferences?
  - What are the feasible consumption bundles?
  - What is (are) the best consumption bundle(s) among all feasible consumption bundles?

Preference Relations

To model individual choices, we first need a preference relation.

- Your preference relation allows you to rank two feasible alternative. For example, if you have two bundles x and y, you preference relation allows you to compare x and y.
- A **consumption bundle** is a vector

$$x = (x_1, x_2, ..., x_n)$$

  - x_i is the amount of the ith good in the bundle
  - each $x_i \geq 0$
  - usually, each good is assumed to be perfectly divisible (1 apple, 1.1 apple, 1.01 apple, etc)

  Example: $x = \{$1kg of apples, 2kg of bananas, 1liter of water$\}$.

- The **objectives of the decision maker** (here, a representative consumer) are summarized in a **preference relation**, that is denoted $\succsim$.

Preference Relations

To model individual choices, we first need a preference relation.

- Your preference relation allows you to rank two feasible alternatives. For example, if you have two bundles x and y, you preference relation allows you to compare x and y.

In the theory of the consumer, preference relations are defined over the set of all consumption bundles.

- Consider all feasible consumption choices.
- We want to equip the consumer with a preference relation defined over all feasible consumption choices.
- **Set of all feasible alternatives** = all feasible consumption bundles, denoted X.
- The **objectives of the decision maker** (here, a representative consumer) are summarized in a **preference relation** defined on X, that is denoted $\succsim$.
- Let x and y be two consumption bundles of the set X. When $x \succsim y$, we say that x is "at least as good" as y (or that x is "weakly preferred" to y).

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Definition

Let $\succsim$ be the (weak) preference relation defined in the set of feasible alternatives X.

- Example: Consider two feasible alternatives x and y. Then, if the consumer with preference relation $\succsim$ (weakly) prefers the bundle x to the bundle y, we write $x \succsim y$.

Definition

Let $\succsim$ be the (weak) preference relation defined in the set of feasible alternatives X.

- Example.

Remark: $\succsim$ is not the same thing as $\geq$. When I write $x \geq y$, I mean that the number or vector x is larger or equal to y. When I write $x \succsim y$, I mean that (i) the consumer prefers x over y or (ii) is indifferent between x and y.

- Let $x = 2$ and $y = 1$ be the quantities of a good consumed by a consumer. Obviously, $2 \geq 1$. Yet, it is not obvious that $x \succsim y$. A consumer does not necessarily prefer more of the good to less of the good. You can think about the good "pollution". In that case, you will certainly have $1 \succsim 2$ while $2 \geq 1$!

Notations

- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}_+$: the set of positive real numbers.
- $\mathbb{R} \times \mathbb{R}$, also denoted $\mathbb{R}^2$: the set of vectors with two components. Example: $v_1 = (1, 2)$, $v_2 = (3, 4)$, $v_3 = (1, 4)$. The vector v_1 has two components; the first component is 1, the second is 2.
- $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, also denoted $\mathbb{R}^3$ is the set of vectors with three components. Example: $v_4 = (1, 2, 1)$.
- and so on. For instance, if I denote by $\mathbb{R}^L$ the set of vector with L components (each being a real number), where L is an integer. $\mathbb{R}^L_+$ is the set of vector with L components (each being a non-negative real number), where L is an integer.

Notations

- Let x and y be two real numbers. Formally, I denote that as follows: $x, y \in X$. For the moment, assume X is $\mathbb{R}^2$. Can you say that v_3 is larger than v_1? The answer is not obvious.
- We need to introduce several definitions for vector inequalities:
  - $>>$ is the "strict inequality" between vectors. Consider two vectors of X^L, $v_5 = (v_{51}, v_{52}, ..., v_{5L})$ and $v_6 = (v_{61}, v_{62}, ..., v_{6L})$. We say that $v_5 >> v_6$ (the vector v_5 is strictly larger than the vector v_6) if and only if $v_{51} > v_{61}$, $v_{52} > v_{62}, ..., v_{5L} > v_{6L}$. In words, when you compare the same components of v_5 and v_6, the component of the former vector is strictly larger than the component of the second vector.
  - $\geqq$ is the "inequality" between vectors. We say that $v_5 \geqq v_6$ (the vector v_5 is larger or equal to the vector v_6) if and only if $v_{51} \geq v_{61}$, $v_{52} \geq v_{62}, ..., v_{5L} \geq v_{6L}$. In words, when you compare the same components of v_5 and v_6, the component of the former vector is larger than or equal to the component of the second vector.

Notations

- Example: $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 1)$, $v_3 = (2, 2, 3)$.
- We have $v_3 >> v_1$ because $2 > 1$ and $3 > 1$.
- We have $v_3 \geqq v_2$ because the second components of v_2 and v_3 are both equal to 2.
- But we don't have $v_3 >> v_2$.

From the preference relation "at least as good as" denoted $\succsim$, we can obtain **two other preference relations**.

- The strict preference relation $\succ$ defined by

$$x \succ y \Leftrightarrow [x \succsim y \text{ but not } y \succsim x]$$

  We say that x is strictly prefered to y
- The indifference relation $\sim$ defined by

$$x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x].$$

  We say that x and y are indifferent

Preference relations: Summary of the Notations

- $\succ$: the strict preference relation. In particular, indifference is ruled out.
- $\succsim$: the "weak" preference relation. In particular, you can be indifferent.
- $\sim$: the indifference relation
- Examples: If you are crazzy about macroeconomics but hate microeconomics, you will have

$$macroeconomics \succ microeconomics.$$

If you weakly prefers macroeconomics to microeconomics, you can either (i) strictly prefer macroeconomics to microeconomics or (ii) be indifferent between macroeconomics and microeconomics. We write that as follows: $macro \succsim micro$.
If you like macroeconomics as much as microeconomics, $macro \sim micro$.

- Preference relations are the basic ingredients of consumer theory. Without a preference relation, the consumer cannot make choices.
- In microeconomics, we are interested in individuals behaving in a rather "rational" way. Usually, we are not interested in "crazzy" or "irrational" choices.
- For this reason, we need to introduce a number of assumptions on the consumer's behaviour. To define "rationality" in a precise way, we need to introduce two definitions:
  - completeness of a preference relation
  - transitivity of a preference relation

The first definition pertains to the completeness of the preference relation.

- The basic idea is that a "rational" consumer must be able to rank **all** available bundles
- Completeness rules out the situation in which a consumer is unable to rank two bundles (note that unable to rank does not mean that you are indifferent – it means that you are unable to make a choice).

Definition

The preference relation $\succsim$ defined on X is **complete** if and only if for every x, y in X, we have

$$x \succsim y \text{ or } y \succsim x \text{ (or both).}$$

This means that the individual is able to rank all feasible bundles.

The second definition pertains to transitivity.

- Transitivity simply captures the idea that if you prefer apples to bananas and bananas to grapefruit, then you should prefer apples over grapefruit.
- This is another behaviour assumption.

Definition

The preference relation $\succsim$ defined on X is **transitive** if and only if for every x, y, z in X, we have

$$[x \succsim y \text{ and } y \succsim z] \Rightarrow x \succsim z.$$

If x is at least as good as y and y at least as good as z, then x is at least as good as z.

In much of microeconomics, individual preferences are assumed to be **rational**. Rationality is defined in a precise way. It is the combination of completeness and transitivity.

Definition

The preference relation $\succsim$ defined on X is **rational** if and only if it is complete and transitive.

- Note that completeness and transitivity may be strong assumptions:
  - hard to evaluate the consequences of all alternatives (completeness)
  - problem of just perceptible preferences. Choice between two gray paints. If paints are almost the same, very difficult. Yet, you certainly have a preference between the darkest and the lighter shade. Violates transitivity.

Desirability Assumption

- We have defined completeness, transitivity and rationality.
- Other behavioural assumption are useful to the analysis.
- They are just assumptions on the behaviour of the economic consumer.
- Desirability assumptions are introduced to have further restrictions on individual preferences.

Desirability Assumption

A first idea of "desirability" is captured by "monotonicity".

- The basic idea is that "more" is "better".
- But what is the meaning of "more"?
- Consider two bundles of good 1 and good 2, $(2, 3)$ and $(1, 10)$. Will you prefer $(2, 3)$ or $(1, 10)$? Everything depends on the preference relation of the consumer.
- Now consider two bundles $(2, 3)$ and $(3, 5)$. Can we say that the consumer prefers $(3, 5)$ to $(2, 3)$? Once again, the answer depends on the consumer preferences. Yet, if the consumer's preference satisfies an additional assumption, called monotonicity, we are able to provide a clear-cut answer.

Desirability Assumption

Definition of Monotonicity:

Definition

The preference relation $\succsim$ defined on X is monotone if and only for x, y in X,

$$y >> x \Longrightarrow y \succ x$$

- We consider two bundles $y = (y_1, ..., y_L)$ and $x = (x_1, ..., x_L)$
- $>>$ denotes the strict inequality between two vectors. Formally,

$$y >> x \Leftrightarrow y_\ell > x_\ell \text{ for every } \ell = 1, ..., L.$$

- Monotonicity: consider a bundle x. Increase the amount of every good in the bundle. Then, you strictly prefer the new bundle to the initial one. Hence, $(3, 5)$ prefered to $(2, 3)$ because there is more of every good in the first bundle.

Desirability Assumption

Before going further, let us introduce a few notations:

- Given two bundles $x = (x_1, ..., x_L)$ and $y = (y_1, ..., y_L)$ of the consumption set X, $\| x - y \|$ is the Euclidian norm of $x - y$. It corresponds to the distance between both bundles. Formally,

$$\|x - y\| = \sqrt{\sum_{\ell=1}^{L} (x_\ell - y_\ell)^2} = \sqrt{(x_1 - y_1)^2 + ... + (x_L - y_L)^2}.$$

  Note that $\sum_{\ell=1}^{\ell=L} x_l$ is defined by $\sum_{\ell=1}^{\ell=L} x_l = x_1 + x_2 + ... + x_L$.
- $\forall$ means "for any". Example: $\forall x \in X$ usually means for any bundle x in the consumption set X.
- $\exists$ means "there is"
- $A \Leftrightarrow B$ means A is satisfied "if and only if" B is satisfied
- $A \Rightarrow B$ means A "implies" B
- If $A \Leftrightarrow B$, then we both have: $A \Rightarrow B$ and $B \Rightarrow A$

Desirability Assumption

- Monotonicity is a strong assumption. In addition, it does not allow you to say whether the consumer prefers the bundle $(2, 5)$ over the bundle $(1, 200)$ because the second bundle has less commodity 1 than the first bundle does.
- Because monocity appears as a strong requirement, a weaker requirement might be preferable.
- Weaker desirability assumption: **local nonsatiation**

Desirability Assumption

A preference relation satisfies local nonsatiation, if for any bundle x the consumer can find another bundle y arbitrarily close to y but preferred to x.

- What do I mean? Let $X = \mathbb{R}^2_+$ (the choice set is the set of vectors with two components, each component being a non-negative real number). Consider a bundle $(2, 3)$ (you have 2 units of good 1 and 3 units of good 2). If $\succsim$ satisfies local non-satiation, then we know that there is at least one bundle y, close to bundle x, which is preferred to x.
- Why is local nonsatiation a less restrictive assumption than monotonicity? Because the bundle y preferred to x does not need to consist of larger quantities of every good in the bundle. In addition, y might consist of less of every good.

Desirability Assumption

Definition

The preference relation $\succsim$ defined on X is locally nonsatiated if and only if for every x in X, there is a bundle y in any neighbourhood of x such that $y \succ x$.

- Illustration
- Formal definition of local nonsatiation: for any $\epsilon > 0$, there is a bundle y with $\|y - x\| \leq \epsilon$ such that $y \succ x$.
- In the formal definition $\|y - x\| \leq \epsilon$ means that the number (or distance) $\|y - x\|$ is less or equal to the number ϵ. In the last part of the formal definition, $y \succ x$ means that the bundle y is "stricly preferred" to the bundle x (this is not equivalent to using $>$!)
- Note that monotonicity implies local nonsatiation. Why?

Indifference Sets

Given the preference relation $\succsim$ defined on X and a consumption bundle x in X, we can define three sets of consumption bundles:

- The **indifference set**: set of bundles which are indifferent to x, i.e.

$$\{y \in X : y \sim x\}$$

- Upper-contour set of x: set of bundles which are preferred or indifferent to the bundle x, i.e.

$$\{y \in X : y \succsim x\}$$

- Lower-contour set of x: set of bundles which are less valued or indifferent to the bundle x, i.e.

$$\{y \in X : x \succsim y\}$$

Partial Summary

- The economic theory of choice begins by describing people's preferences. This is why we first focused on preference relations. Why? Because we want to be able to describe the choices made by a consumer.
- Because we want to build a **model** of the consumer behaviour, we have considered a few assumptions regarding the behaviour of the consumer.

1. First, we have considered that the consumer is able to rank **all available bundles** (**completeness**).

Definition

A consumer is rational if his preference relation is complete and transitive.

Partial Summary

- The economic theory of choice begins by describing people's preferences. This is why we focused last week on preference relations. Why? Because we want to be able to describe the choices made by a consumer.
- Because we want to build a **model** of the consumer behaviour, we have considered a few assumptions regarding the behaviour of the consumer.

1. First, we have considered that the consumer is able to rank **all available bundles** (**completeness**).
2. Second, we have considered that the consumer's choices satisfy a consistency axiom: if x is preferred to y and y to z, then x must also be preferred to z (**transitivity**).

Definition

A consumer is rational if his preference relation is complete and transitive.

Indifference Sets

Implication of local nonsatiation for indifference sets?

- Consider a "thick" indifference curve. Graph.
- Is locally nonsatiation satisfied?

Convex Set

The following definition is useful before going further.

Definition

A set $S \subseteq \mathbb{R}^n$ is convex if and only if

$$\forall (\mathbf{x}, \mathbf{y}) \in S^2 : [\mathbf{x}, \mathbf{y}] \in S$$

where

$$[\mathbf{x}, \mathbf{y}] = \{\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} : \lambda \in [0, 1]\}$$

- Graph of a convex set
- Examples of non-convex sets

Convex Preference Relation

Definition

The preference relation $\succsim$ defined on X is convex if and only if for every x in X, the upper-contour set is convex.

- Graph of a convex preference relation
- Example of a non-convex preference relation
- Intrepretation: basic inclination of economic agents to diversification. For every x and y in X, an individual prefers $\lambda x + (1 - \lambda) y$, with $\lambda \in [0, 1]$, to x or y.

A Few Words on Convexity (Clarification)

- Given a preference relation $\succsim$ defined on a choice set X and any bundle x in X, the **upper-contour set of** x is defined as the set of bundles which are indifferent or superior to x. Formally, the upper-contour set of x is defined as

$$\{y \in X : y \succsim x\}.$$

- The preference relation $\succsim$ defined on the choice set X is said to be **convex** if and only if for any bundle x in the choice set X, the upper-countour set of x is convex.
- Interpretation: consider a bundle y on the indifference set of the bundle x. Then, any bundle obtained as a combination of x and y is preferred to x and y. Formally, this means that any bundle $\lambda x + (1 - \lambda) y$ where λ is any number between 0 and 1 is preferred to x and y. In words, a consumer with convex preferences **prefers variety** in his consumption choices.

In economics, we often describe preference relation by means of a utility function. Consider a utility function $u(x) = u(x_1, ..., x_n)$. It assigns a number $u(x)$ to every element x of the choice set X

Definition

The utility function $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succsim$ on X if and only if

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

- Consider x weakly preferred to y. Compute $u(x)$ and $u(y)$. Then, $u(x)$ is larger or equal to $u(y)$.
- Assume u represents $\succsim$. Is u unique? Why?

Why is rationality a useful assumption from the technical viewpoint?

Theorem

A preference relation $\succsim$ can be represented by a utility function only if it is rational

Proof: we show that if u is a utility function which represents $\succsim$, then $\succsim$ must be complete and transitive.

- because u is a function defined on X, for every x, y in X, we have either $u(x) \geq u(y)$ or $u(x) \leq u(y)$. Because u represents $\succsim$, this implies $x \succsim y$ or $y \succsim x$. Hence, $\succsim$ is complete.
- suppose $x \succsim y$ and $y \succsim z$. Because u represents $\succsim$, then $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Hence, $u(x) \geq u(z)$. This implies $x \succsim z$. Hence, $\succsim$ is transitive.

Consumption Sets

- Consumption choices are typically limited by physical constraints
- Example: you cannot consume a negative amount of a commodity
- Other examples: graphs

Consumption Sets

- In the examples: the constraints are physical in a very literal sense.
- Yet, the constraints can also be institutional. Example: a law requiring that no one works more than 12 hours a day.

Consumption Sets

From now on, we consider the simplest sort of consumption set:

$$X = \mathbb{R}^L_+ = \left\{ x \in \mathbb{R}^L : x_\ell \geq 0 \text{ for } \ell = 1, ..., L \right\}$$

This is the set of non-negative bundles

- Note that if a commodity is water pollution, we can define x_l as "absence of water pollution" (we do that because when defining an economic "good" we have in mind that "more is better")
- A special feature of the set $X = \mathbb{R}^L_+$ is that it is convex.

Competitive Budgets

- Up to now: physical or institution constraints to define consumption sets
- In addition: individual choices are limited by monetary constraints

Definition

The competitive budget of a consumer is the set of all feasible consumption bundles the consumer can afford.

Competitive Budgets

To get a more precise definition, we need to introduce:

- **Commodity prices** $p = (p_1, p_2, ..., p_L)$.
  For simplicity, we assume that $p_\ell > 0$ for every $\ell = 1, ..., L$
- **Individual wealth** w
  For simplicity, we assume that $w > 0$ is exogeneously given (can be endogenized $\rightarrow$ see leisure/consumption choice)
- **Price-taking assumption**: the consumer has no influence on prices

Competitive Budgets

Hence, a competitive budget can be defined as follows.

Definition

The competitive budget $B(p, w)$ of a consumer is the set of all feasible consumption bundles the consumer can afford, i.e.,

$$B(p, w) = \left\{ x \in X : \sum_{\ell=1}^{L} p_\ell x_\ell \leq w \right\}.$$

Competitive Budgets

- Competitive budget $B(p, w)$ of a consumer:

$$B(p,w) = \left\{ x \in X : \sum_{\ell=1}^{L} p_\ell x_\ell \leq w \right\}.$$

- **Budget constraint** of the consumer: the boundary of the set $B(p, w)$, i.e. the set

$$\left\{ x \in X : \sum_{\ell=1}^{L} p_\ell x_\ell = w \right\}.$$

Voluntary Trades and Indifference Curves

- We previously have considered indifference curves which are decreasing and convex.
- Consider the indifference curve drawn in class. How many additional units of good 1 should the consumer receive to keep the same utility level if he loses 1 unit of good 2?
- The answer depends on the initial consumption bundle you look at on the indifference curve.
  - For instance, at $(1,7)$, the consumer must receive 2 additional units of good 1 to keep utility constant while losing 1 unit of good 2.
  - At $(3,6)$, the consumer must receive 3 additional units of good 1 to compensate for a 1-unit loss in good 2 while keeping utility constant.
  - At $(6,5)$, the consumer must receive 6 additional units of good 1 to compensate for a 1-unit loss in good 2 while keeping utility constant.

Voluntary Trades and Indifference Curves

- Let $x = (3, 6)$ and $y = (6, 5)$. We can compute the slope of the line segment $[x, y]$ as

$$\text{slope of } [x, y] = \frac{\text{Change in good 2}}{\text{Change in good 1}} = \frac{6-5}{3-6} = -\frac{1}{3}.$$

- Interpretation: The consumer needs to receive 3 units of good 1 to compensate for a 1-unit loss in good 2 and keep the same utility level.
- The ratio $-1/3$ is an approximation of the slope of the indifference curve in the neighbourhood of bundle x.

Marginal Rate of Substitution

- Consider the indifference curve through bundle x. Then, consider an **infinitesimal decrease** in the amount of good 1 available to the consumer. How many additional units of good 2 should be given to the consumer so that he keeps the same utility level as before?
- Call y the new bundle. Because y tends to zero, the line segment $[x, y]$ is the **tangent** to the indifference curve through bundle x.
- Assume the slope of the tangent is $-1/3$ at bundle x. This means that the consumer keeps the same utility level when he trades 3 units of good 1 against 1 unit of good 2. In other worlds, the **"exchange rate"** between good 2 and good 1 is equal to $1/3$ (in the neighbourhood of bundle x).
- This exchange rate is called **marginal rate of substitution of good 2 for good 1 at bundle** x.

Marginal Rate of Substitution

Definition

The marginal rate of substitution (MRS) of good 2 for good 1 at bundle x corresponds to the (absolute value of the) slope of the indifference curve through x at bundle x (provided the indifference curve is differentiable at x).

- When preferences are convex, the indifference curves are decreasing and convex.
- Consider any indifference curve. Its slope increases when the quantity of good 1 is increased. Because this slope is negative, its absolute value decreases when the quantity of good 1 is increased. Consequently, **when preferences are convex, the marginal rate of substitution of good 2 for good 1 is decreasing** (in absolute value) **along any indifference curve.**
- The basic idea captured by the decrease in the MRS is that **balance in consumption is desirable.**

Indifference Curve Maps

- Consider a rational consumer with monotonic and convex preferences
  - Monotonicity ⇒ indifference sets are not thick
  - Convexity ⇒ the upper-contour set of any indifference set is convex ⇒ balance in consumption is desirable
  - Rationality ⇒ the consumer's preference relation is complete and transitive

Indifference Maps

- Because of completeness, the consumer is able to rank all bundles in the choice set $\Rightarrow$ **there are indifference curves through every bundle**
- Because of transitivity, indifference curves cannot cross $\Rightarrow$ **there is one and only one indifference curve through every bundle**
- We call **indifference curve map** the set of indifference curves through the bundles in the choice set. The indifference map is a representation of the consumer's preferences.

Indifference Curve Maps for Particular Preferences

- Assume the consumer can consume two goods, called good 1 and good 2, in non-negative amounts. Then, the consumer's choice set is $X = \mathbb{R}_+ \times \mathbb{R}_+$ (or $X = \mathbb{R}_+^2$).
- Example 1: **Assume good** 2 **is useless for the consumer.**
  - As a result, the utility level of the consumer is independent of the amount of good 2 in his consumption bundle
  - What is the shape of the indifference curve map?

Indifference Curve Maps for Particular Preferences

- Example 2: **Assume good** 2 **is an economic bad (such as pollution)**
  - As a result, the utility level of the consumer decreases when the amount of good 2 increases.
  - To keep the same utility level, the consumer must receive additional units of good 1 when there are more units of good 2 in his consumption bundle.
  - What is the shape of the indifference curve map?

Indifference Curve Maps for Particular Preferences

- Example 3: **Assume good 1 and good 2 are perfect substitutes.**
  - The consumer is indifferent between α unit of good 1 and α unit of good 2 (where α is any positive number).
  - Hence, the marginal rate of substitution of good 2 for good 1 is equal to 1 at any bundle.
  - The implication is that indifference curves are straight lines with slope equal to -1.
  - Shape of the indifference curve map?

Indifference Curve Maps for Particular Preferences

- Example 4: **Assume good** 1 **and good** 2 **are perfect complements.**
  - Example: in a pair of shoes, right shoe and left shoe go together.
  - If you have 2 right shoes but 1 left shoe, your utility is certainly the same as if you have only 1 right shoe.
  - Shape of the indifference curve map?

What the consumer can (and cannot) afford. The Budget Set

- We consider a rational consumer making choice in the choice set X.
- If the consumer is able to rank all bundles, he cannot afford everything he wants.
- In particular, the choices are constrained by the consumer's income.
- Today, we consider that the income of the consumer is exogenously given and that the consumer cannot save income to consume goods tomorrow (These cases will be considered later, cf. labour/consumption choices and intertemporal choices).

Budget Set of the Consumer, Definition

Definition

The budget set of the consumer is the set of all bundles that the consumer can afford.

What the consumer can afford depends on:

- his income $\mathcal{I}$
- the prices of the goods

Budget Constraint Algebra, An Example

- Assume the consumer can consume two goods, called good 1 and good 2, in non-negative amounts x_1 and x_2 respectively. Then, the consumer's choice set is $X = \mathbb{R}_+ \times \mathbb{R}_+$ (or $X = \mathbb{R}^2_+$).
- Let $\mathcal{I}$ be the exogenous income of the consumer.
- Let p_1 and p_2 be the prices of one unit of good 1 and good 2 respectively.
- Assume prices do not vary with quantities bought.

Budget Constraint Algebra, An Example

- Consider any bundle $x = (x_1, x_2)$ consisting of x_1 units of good 1 and x_2 units of good 2.
- Bundle x costs $p_1x_1 + p_2x_2$
- Bundle x is affordable if and only if its costs is less or equal to $\mathcal{I}$, i.e. if and only if

$$p_1x_1 + p_2x_2 \leq \mathcal{I}$$

- The budget set of the consumer is thus formally defined as

$$\{x = (x_1, x_2) \in X : p_1x_1 + p_2x_2 \leq \mathcal{I}\}.$$

In words, the budget set consists of all bundles $x = (x_1, x_2)$ in the choice set X whose cost $p_1x_1 + p_2x_2$ is less or equal to $\mathcal{I}$.

Budget Constraint Algebra, An Example

- The budget line of the consumer is formally defined as

$$\{x = (x_1, x_2) \in X : p_1 x_1 + p_2 x_2 = \mathcal{I}\}.$$

  It consists of all bundles in the choice set which exactly cost $\mathcal{I}$. To buy one of these bundles, the consumer must spend his whole income $\mathcal{I}$.

- What is the shape of the budget line?
  - Consider the consumption space (good 1, good 2).
  - Then, the budget line has equation

$$x_2 = \frac{\mathcal{I}}{p_2} - \frac{p_1}{p_2} x_1$$

  It is the line with intercept $\mathcal{I}/p_2$ and slope $-p_1/p_2$.

Comparative Statics of the Budget Constraint

- The budget constraint has equation

$$x_2 = \frac{\mathcal{I}}{p_2} - \frac{p_1}{p_2} x_1$$

- Impact of a change in $\mathcal{I}$?
- Impact of a change in p_1? in p_2?

More Complicated Budget Constraint

- Prices can depend on quantities bought by the consumer. For instance, there may be a discount above a threshold.
- In that case, the budget constraint will have a kink at the threshold level, corresponding to a change in the price ratio.
- Graph of such a budget constraint [Add indifference curves and illustrate bunching and holes].

The Consumer's Choice

What is the **best choice of a rational consumer?**

- From preferences, one gets the indifference curve map of the consumer.
- Assume there are L goods and prices are independent on quantities. Given the consumer's exogenous income $\mathcal{I}$ and the prices p_1, p_2,...,p_L of good 1, good 2,..., good L, one gets the budget constraint

$$\sum_{\ell=1}^{L} p_\ell x_\ell \leq \mathcal{I}$$

and the budget line

$$\sum_{\ell=1}^{L} p_\ell x_\ell = \mathcal{I}.$$

- From now on, assume $L = 2$ (so that graphical illustrations are possible).

The Utility Maximization Programme

Problem

Given income $\mathcal{I}$ and the price vector p, choose the **affordable** *consumption bundle x in the consumption set X which is* **preferred over all other affordable bundles.**

If individual preferences are represented by the utility function $u(x_1, x_2)$, the utility maximisation programme can be stated as follows:

Problem

Given income $\mathcal{I}$ and the price vector $p = (p_1, p_2)$, choose the consumption bundle $x = (x_1, x_2)$ in X for which

$$p_1 x_1 + p_2 x_2 \leq \mathcal{I}$$

and utility $u(x_1, x_2)$ is maximum.

Characterization of the Best Bundle

- Consider convex preferences
- Assume indifference curves do not cross the x_1-axis and the x_2-axis
- Then, the best choice of the consumer must be such that $x_1 > 0$ and $x_2 > 0$
- In the optimum,

$$MRS_{12}(x_1, x_2) = \frac{p_1}{p_2}$$

  To see that, graphical argument

  - First, explain why x must be on the budget line
  - Second, explain that an equilibrium cannot be at a point where the MRS is not equal to the price ratio

Illustration: Cases in which the above characterization does not apply

Show the consumer's choice for:

- a useless good
- an economic bad
- perfect substitutes
- perfect complements

Examples

- Show the consumer's choice for a more complicated budget constraint. Possibility of multiple choices
- Show the consumer's choice for non-convex preferences: possibility of corner solutions.

Characterization (Formal argument)

- Consider **rational, monotone and convex preferences**
- Assume indifference curves do not cross the x_1-axis and the x_2-axis
- Let $u(x_1, x_2)$ represent the consumer's preferences. Assume u is twice continuously differentiable, concave, with $u_1' > 0$ and $u_2' > 0$
- Utility maximisation programme: find $(x_1, x_2) >> 0$ which maximises $u(x_1, x_2)$ subject to the budget constraint $p_1 x_1 + p_2 x_2 \leq \mathcal{I}$.
- Write the Lagragian

$$L = u(x_1, x_2) + \lambda [\mathcal{I} - (p_1 x_1 + p_2 x_2)]$$

and then the first-order conditions.

Characterization (Formal argument)

The marginal rate of subtitution between two goods at a given bundle is equal to the absolute value of the slope of indifference curve at this bundle. By definition, the indifference curve of level k has equation

$$u\left(x_1, x_2\right) = k.$$

This equation implicitely defines x_2 as a function of x_1 (and of the constant k). So, we write: $x_2 = x_2\left(x_1\right)$ (with a slight abuse of notation). By differentiation,

$$\frac{\partial u\left(x_1, x_2\left(x_1\right)\right)}{\partial x_1} dx_1 + \frac{\partial u\left(x_1, x_2\left(x_1\right)\right)}{\partial x_2} dx_2\left(x_1\right) = 0.$$

Hence,

$$\frac{dx_2\left(x_1\right)}{dx_1} = -\frac{\frac{\partial u(x_1, x_2(x_1))}{\partial x_1}}{\frac{\partial u(x_1, x_2(x_1))}{\partial x_2}}.$$

Characterization (Formal argument)

Finally,

$$MRS_{12}\left(x_1, x_2\right) = \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}.$$

Interpretation in terms of marginal utility: if Δ denotes a change,

$$MRS_{12}\left(x_1, x_2\right) = \frac{\Delta \text{ in utility from consuming 1 additional unit of good 1}}{\Delta \text{ in utility from consuming 1 additional unit of good 2}}.$$

Clear interpretation as a psychological rate of exchange between two goods.

Characterization (Formal argument)

The Lagrangian is $L = u(x_1, x_2) + \lambda [\mathcal{I} - (p_1 x_1 + p_2 x_2)]$.

There are two choice variables: x_1 and x_2.

The first-order (necessary) conditions for an (interior) maximum are

$$\frac{\partial L}{\partial x_1} = 0 \text{ and } \frac{\partial L}{\partial x_2} = 0.$$

Because

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \Leftrightarrow \lambda = \frac{\partial u(x_1, x_2)/\partial x_1}{p_1}, \\ \frac{\partial L}{\partial x_2} &= \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \Leftrightarrow \frac{\partial u(x_1, x_2)}{\partial x_2} - \frac{\partial u(x_1, x_2)}{\partial x_1} \frac{p_2}{p_1} = 0, \end{aligned}$$

a necessary condition for a maximum is

$$\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{p_1}{p_2} \Leftrightarrow MRS_{12}(x_1, x_2) = \frac{p_1}{p_2}.$$

Characterization (Formal argument)

Alternatively, at the optimum,

$$\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{p_1} = \frac{\frac{\partial u(x_1,x_2)}{\partial x_2}}{p_2}.$$

- Intepretation: The ratio of the extra utility from consuming one more unit of a good to its price should be the same for each good. Or: Each good should provide the same extra utility per dollar spent.
- Assume this is not the case. For example, assume that an extra dollar in good 1 yields more utility than an extra dollar in good 2. Then, good 1 is locally a more effective way of "buying" utility. So, when you should substitute good 2 for good 1, you increase your utility level. Consequently, the initial situation was not optimum.

Cobb-Douglas Example

- Assume a consumer can comsume two goods, 1 and 2, in quantities x_1 and x_2.
- His preference relation is represented by the Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

  with $0 < \alpha < 1$.
- Write the consumer's optimization programme.
- Explain why he won't choose $x_1 = 0$ and/or $x_2 = 0$ in the optimum. Hence, in the optimum, $x_1 > 0$ and $x_2 > 0$.
- Use the necessary conditions for an interior maximum or the characterization

$$MRS_{12}(x_1, x_2) \equiv \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{p_1}{p_2}.$$

Cobb-Douglas Example

- Alternatively, use a Lagrangian to find

$$MRS_{12}\left(x_1, x_2\right) \equiv \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{p_1}{p_2}.$$

- In the optimum,

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{\left(1-\alpha\right) x_1^{\alpha} x_2^{-\alpha}} = \frac{p_1}{p_2} \Leftrightarrow x_2 = \frac{p_1}{p_2}\frac{\left(1-\alpha\right)}{\alpha}x_1.$$

Cobb-Douglas Example

- Then, use the budget constraint $p_1x_1 + p_2x_2 = \mathcal{I}$ to get

$$
\begin{aligned}
p_1x_1 + p_2x_2 &= I \Leftrightarrow p_1x_1 + p_2\frac{p_1}{p_2}\frac{(1-\alpha)}{\alpha}x_1 = \mathcal{I} \\
&\Leftrightarrow p_1\left[1 + \frac{(1-\alpha)}{\alpha}\right]x_1 = \mathcal{I} \\
&\Leftrightarrow x_1 = \frac{\alpha\mathcal{I}}{p_1}.
\end{aligned}
$$

- Interpretation of α and $1-\alpha$.